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On the Propagation of Round-Off Errors in the Numerical Treatment of the Wave Equation

By Arnold N. Lowan

Abstract. An upper bound of the norm of the error vector after n time steps is $\frac{1}{2}(n+1)(n+2) \| \boldsymbol{\delta}^* \|$. For the explicit scheme $\delta^* = \| \boldsymbol{\delta}^* \| = 3 \times \frac{1}{2} \times 10^{-p}$ where p is the number of decimals carried in the computations. For the implicit scheme $\delta^* = \| \boldsymbol{\delta}^* \|$ is an upper bound of the errors which arise both from using approximations to A^{-1} and $A^{-1}B$ in the determination of $\mathbf{u}_{k+1}$ from equation (6*) and from rounding off the values of the products and quotients involved in the computation of the components of $\mathbf{u}_{k+1}$.

Consider the numerical treatment of the differential equation of wave motion

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 \leqq x \leqq a, \quad t > 0 \tag{1}$$

the solution of which is required to satisfy the following initial and boundary conditions

$$u(x, 0) = f(x) \tag{2}$$

$$u_t(x, 0) = g(x) \tag{3}$$

$$u(0, t) = u(a, t) = 0. \tag{4}$$

With the differential equation (1) we will associate either of the following two difference analogs [1]

$$u_{h,k+1} - 2u_{h,k} + u_{h,k-1} = R^2(u_{h-1,k} - 2u_{h,k} + u_{h+1,k}) \tag{5}$$

$$\begin{aligned} u_{h,k+1} - 2u_{h,k} + u_{h,k-1} = \frac{R^2}{2} (u_{h-1,k+1} &- 2u_{h,k+1} + u_{h+1,k+1} \\ &+ u_{h-1,k-1} - 2u_{h,k-1} + u_{h+1,k-1}) \end{aligned} \tag{6}$$

where $R = c\Delta t/\Delta x$ and $u_{h,k} = u(h\Delta x, k\Delta t)$ with $(M + 1)\Delta x = a$.

The difference counterpart of (3) will be taken in the form

$$\frac{u_{h,1} - u_{h,0}}{\Delta t} = g(h\Delta x);$$

whence

$$u_{h,1} = u_{h,0} + g(h\Delta x)\Delta t = f(h\Delta x) + g(h\Delta x)\Delta t. \tag{7}$$

The difference equations (5) and (6) may be written in the compact forms

$$\mathbf{u}_{k+1} = A\mathbf{u}_k - \mathbf{u}_{k-1} \tag{5*}$$

$$A\mathbf{u}_{k+1} = 4\mathbf{u}_k + B\mathbf{u}_{k-1}. \tag{6*}$$

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In (5*) A is a tridiagonal matrix whose elements on the principal diagonal are $= 2(1 - R^2)$ and whose elements off the principal diagonal are $= R^2$ and $\mathbf{u}_k$ is the vector whose components are the values of $u(x, t)$ at time $t = k\Delta t$ at the lattice points $x = h\Delta x, h = 1, 2, 3 \cdots$. In (6*) A is a tridiagonal matrix whose elements on the principal diagonal are $= 2(1 + R^2)$ while the elements off the principal diagonal are $= -R^2$ and B is a tridiagonal matrix whose elements on the principal diagonal are $= -2(1 + R^2)$ while the elements off the principal diagonal are $= R^2$.

Consider first the explicit difference scheme (5*). Since both $\mathbf{u}_0$ and $\mathbf{u}_1$ are known, (5*) will yield in succession $\mathbf{u}_2, \mathbf{u}_3 \cdots$. Specifically,

$$\left\{\begin{array}{l}\mathbf{u}_2 = A\mathbf{u}_1 - \mathbf{u}_0\\ \mathbf{u}_3 = A\mathbf{u}_2 - \mathbf{u}_1\\ \vdots\\ \mathbf{u}_n = A\mathbf{u}_{n-1} - \mathbf{u}_{n-2}.\end{array}\right. \tag{8}$$

It is reasonable to assume that the components of $\mathbf{u}_0$ are exact while those of $\mathbf{u}_1$, obtained from (7), have been rounded off to the number of decimal places to be carried in the computations. Let $\mathbf{u}_1^*$ denote the vector whose components are the rounded off values of the components of $\mathbf{u}_1$. It is then easily seen that we introduce two types of errors in the evaluation of $\mathbf{u}_2$. A first error is due to using $\mathbf{u}_1^*$ in lieu of $\mathbf{u}_1$. A second error is introduced as a result of rounding off of the values of the products involved in the expression of $u_{h,k+1}$ obtained from (5) to the number of decimal places carried in the computations. Thus, in lieu of the exact vector $\mathbf{u}_2$, the first step in the sequence of operations (8) yields the vector $\mathbf{u}_2^* = A\mathbf{u}_1^* - \mathbf{u}_0 + \boldsymbol{\delta}_2$ where $\boldsymbol{\delta}_2$ is the error vector whose components are the round-off errors just discussed. Similarly, error vectors are introduced in each of the successive steps in the sequence of operations (8). Thus

$$\left\{\begin{array}{l}\mathbf{u}_2^* = A\mathbf{u}_1^* - \mathbf{u}_0 + \boldsymbol{\delta}_2\\ \mathbf{u}_3^* = A\mathbf{u}_2^* - \mathbf{u}_1^* + \boldsymbol{\delta}_3\\ \vdots\\ \mathbf{u}_n^* = A\mathbf{u}_{n-1}^* - \mathbf{u}_{n-2}^* + \boldsymbol{\delta}_n.\end{array}\right. \tag{9}$$

If we put

$$\mathbf{E}_n = \mathbf{u}_n^* - \mathbf{u}_n \tag{10}$$

then from (8) and (9) it follows that

$$\mathbf{E}_n = A\mathbf{E}_{n-1} - \mathbf{E}_{n-2} + \boldsymbol{\delta}_n. \tag{11}$$

In entirely similar manner it may be shown that the counterpart of (11) for the implicit scheme (6*) is

$$\mathbf{E}_n = 4A^{-1}\mathbf{E}_{n-1} + A^{-1}B\mathbf{E}_{n-2} + \boldsymbol{\delta}_n. \tag{12}$$

There is, however an important distinction between (11) and (12); whereas in (11) the components of $\boldsymbol{\delta}_n$ are round-off errors as above explained, in (12) the components of $\boldsymbol{\delta}_n$ are the aggregate of the errors arising both from using approximations to A^{-1} and $A^{-1}B$ in the determination of $\mathbf{u}_{k+1}$ and the round-off errors

introduced as a result of rounding-off the values of the products and quotients involved in the computation of the components of $\mathbf{u}_{t+1}$.

The error equations (11) and (12) are of the form

$$\mathbf{E}_n = M\mathbf{E}_{n-1} + N\mathbf{E}_{n-2} + \mathbf{\delta}_n . \tag{13}$$

If in (13) we put in succession $n = 2, 3, 4, \cdots$ and write $\mathbf{\delta}_1$ for $\mathbf{E}_1$, it may be shown by induction that

$$\mathbf{E}_n = P_{n-1}(M, N)\mathbf{\delta}_1 + P_{n-2}(M, N)\mathbf{\delta}_2 + \cdots \mathbf{\delta}_n \tag{14}$$

or

$$\mathbf{E}_n = \sum_{p=0}^{n} P_p(M, N)\mathbf{\delta}_{n-p} \tag{14*}$$

where

$$P_n(M, N) = M^n + C^1_{n-1}M^{n-2}N + C^2_{n-2}M^{n-4}N^2 + \cdots C^s_{n-s}M^{n-s}N^s + \cdots \tag{15}$$

or

$$P_n(M, N) = \sum_{s=0}^{(n/2)} C^s_{n-s}M^{n-2s}N^s \tag{15*}$$

where $(n/2)$ denotes the largest integer in $n/2$, where $C_n^{\,0} = 1$ and $C_m^{\,n}$ denote the binomial coefficient $m(m-1)(m-2)\cdots(m-n+1)/n!$.

We shall prove that if M and N have the same eigenvectors, then

$$\| P_p(M, N)\mathbf{\delta}_{n-p} \| \leqq \| \mathbf{\delta}_{n-p} \| \cdot (p + 1) \tag{16}$$

where for any M-dimensional vector $\mathbf{\phi}$, its norm $\| \mathbf{\phi} \|$ is defined by

$$\| \mathbf{\phi} \| = \sqrt{(\mathbf{\phi}, \mathbf{\phi})} = \sqrt{\frac{1}{M}\sum_{h=1}^{M} (\phi_h)^2} \tag{17}$$

the ϕ_h's being the components of $\mathbf{\phi}$,
provided that the roots of the quadratic equation

$$x^2 - \lambda_r x - \mu_r = 0 \tag{18}$$

where the λ_r's and μ_r's, the eigenvalues of M and N respectively, are either numerically equal to or smaller than unity (if real) or have a modulus equal to or smaller than unity (if complex). Indeed, let

$$\mathbf{\delta}_{n-p} = \sum_{r=1}^{M} \alpha_r^{(n-p)}\mathbf{w}_r \tag{19}$$

where the $\mathbf{w}_r$'s are the normalized eigenvectors of the matrices M and N. From (19) and (14*) we get

$$P_p(M, N)\mathbf{\delta}_{n-p} = \sum_{s=0}^{(p/2)} \sum_{r=1}^{M} C^s_{p-s}\, \alpha_r^{(n-p)} M^{p-2s}N^s\mathbf{w}_r . \tag{20}$$

But

$$M^{p-2s}N^s\,\mathbf{w}_r = M^{p-2s}\mu_r{}^s\mathbf{w}_r = \lambda_r^{p-2s}\mu_r{}^s\mathbf{w}_r\,;$$

whence

$$(21)\qquad \begin{aligned} P_p(M,N)\boldsymbol{\delta}_{n-p} &= \sum_{r=1}^{M}\alpha^{(n-p)}\mathbf{w}_r \sum_{s=0}^{(p/2)} C_{p-s}^{s}\,\lambda_r^{p-2s}\mu_r{}^s \\ &= \sum_{r=1}^{M}\beta_r(p)\alpha_r^{(n-p)}\mathbf{w}_r \qquad \text{(say).}\end{aligned}$$

It may be proved by induction that

$$(22)\qquad \beta_r(p) = \sum_{s=0}^{(p/2)} C_{p-s}^{s}\,\lambda_r^{p-2s}\mu_r{}^s = \frac{x_{1,r}^{p+1} - x_{2,r}^{p+1}}{x_{1,r} - x_{2,r}} = \sum_{\sigma=0}^{p} x_{1,r}^{\sigma}\,x_{2,r}^{p-\sigma}$$

where $x_{1,r}$ and $x_{2,r}$ are the roots of the quadratic equation (18). From (22) it is clear that if these roots are numerically smaller than unity then

$$(23)\qquad |\beta_r(p)| < p + 1;$$

and furthermore $\beta_r(p) \to 0$ as $p \to \infty$. In view of (23), (21) yields

$$\| P_p(M,N)\boldsymbol{\delta}_{n-p} \| = \sqrt{\sum_{r=1}^{M}[\beta_r(p)]^2[\alpha_r^{(n-p)}]^2} \leqq (p+1)\sqrt{\sum_{r=1}^{M}[\alpha_r^{(n-p)}]^2}\,;$$

or

$$(16)\qquad \| P_p(M,N)\boldsymbol{\delta}_{n-p} \| \leqq (p+1)\| \boldsymbol{\delta}_{n-p} \|.$$

From (14*) the Minkowski inequality yields

$$(24)\qquad \| \mathbf{E}_n \| \leqq \sum_{p=0}^{n} \| P_p(M,N)\boldsymbol{\delta}_{n-p} \|,$$

whence, in view of (16)

$$(25)\qquad \| \mathbf{E}_n \| \leqq \sum_{p=0}^{n}(p+1)\| \boldsymbol{\delta}_{n-p} \|\,;$$

and a fortiori

$$(25^*)\qquad \| \mathbf{E}_n \| \leqq \| \boldsymbol{\delta}^* \| \sum_{p=0}^{n}(p+1) = \frac{(n+1)(n+2)}{2}\| \boldsymbol{\delta}^* \|,$$

where $\| \boldsymbol{\delta}^* \|$ is the largest of the sequence $\| \boldsymbol{\delta}_1 \|, \| \boldsymbol{\delta}_2 \| \cdots \| \boldsymbol{\delta}_n \|$. If δ^* denotes an upper bound of the components of all the vectors $\boldsymbol{\delta}_p$, it is readily seen that

$$\| \boldsymbol{\delta}^* \| \leqq \delta^*.$$

Furthermore, since

$$\| \mathbf{E}_n \| = \left\{\frac{1}{M}\sum_{h=1}^{M}(E_{nh})^2\right\}^{1/2}$$

where the E_{nh}'s are the components of $\mathbf{E}_n$, it is clear that the maximum of any of the components is obtained by assuming that all but one of the components are = 0.

Calling the maximum value of the components E_n^* we finally get

(26) $$E_n^* \leq \tfrac{1}{2}(n+1)(n+2)\sqrt{M}\delta^*.$$

The second member of (26) is an upper bound of the round-off errors for both the explicit analog (5) and the implicit analog (6).

In the case of the explicit scheme (5) the matrix M of (13) is the matrix A appropriate to (5) while the matrix N of (13) is $= -I$ where I is the $M \times M$ identity matrix. The eigenvalues of A are known [2] to be

(27) $$\lambda_r = 2 - 4R^2 \cos \frac{r\pi}{2(M+1)}$$

The eigenvalues of $-I$ are clearly $= -1$. Thus the quadratic equation (18) becomes

(28) $$x^2 - \left[2 - 4R^2 \cos \frac{r\pi}{2(M+1)}\right] x + 1 = 0.$$

It is clear that if the roots of (28) were real, one would have to be larger than unity, since the products of the roots is $= 1$. Under these conditions $\beta_r(p)$ as defined in (22) would not be bounded as $p \to \infty$ and the difference scheme (5) could not be stable. Thus the roots of (28) must be complex, in which case the modulus of the roots is $= 1$ and $\beta_r(p) \leq p + 1$.

An upper bound of the round-off errors after n time steps is then given by

$$E_n^* = \tfrac{1}{2}(n+1)(n+2)\sqrt{M}\delta^*$$

where $\delta^* = 3 \times \frac{1}{2} \times 10^{-p}$ if the computations are carried to p decimal places. In the case of the implicit scheme (6) matrices M and N of (13) are A^{-1} and $A^{-1}B$ respectively where the matrices A and B appropriate to (6) have been defined earlier.

It can be easily shown that the matrices A^{-1} and $A^{-1}B$ have the same eigenvectors, as required in the above developments [2, p. 20], and that their eigenvalues are

(29) $$\lambda_r = 2\Big/\left(1 + 2R^2 \cos^2 \frac{r\pi}{2(M+1)}\right); \qquad \mu_r = -1$$

Thus the quadratic equation (18) becomes

(30) $$x^2 - \frac{2}{1 + 2R^2 \cos^2 \dfrac{r\pi}{2(M+1)}} x + 1 = 0.$$

Clearly the roots of (30) must be complex. This leads to the condition

$$1/\{1 + 2R^2 \cos [r\pi/2(M+1)]\} < 1$$

which is evidently satisfied for any value of R. Thus the difference scheme (6) is unconditionally stable. Furthermore, and for the same reason as above,

$$\beta_r(p) \leq p + 1.$$

An upper bound of the round-off errors after n time steps is, therefore, once more

given by

$$E^* \leqq \tfrac{1}{2}(n+1)(n+2)\sqrt{M}\delta^*. \tag{26*}$$

In this case, however, as previously mentioned δ^* is an upper bound of the errors which arise both from the use of approximations to A^{-1} and $A^{-1}B$ in lieu of exact matrices and from the process of rounding-off the values of the products and quotients involved in the evaluation of the components of $\mathbf{u}_{k+1}$. Clearly δ^* depends on the specific scheme for solving the system of equations (6) with $h = 1, 2, 3, \cdots M$ for the $u_{h,k+1}$'s.

In order to estimate δ^* for the implicit scheme we note that the counterpart of the typical equation (9) is

$$\mathbf{u}_k^* = 4A^{-1}\mathbf{u}_{k-1}^* + A^{-1}B\mathbf{u}_{k-2}^* + \boldsymbol{\delta}_k$$

whence

$$A\mathbf{u}_k^* = 4\mathbf{u}_{k-1}^* + B\mathbf{u}_{k-2}^* + A\boldsymbol{\delta}_k.$$

Let $\mathbf{R}_k$ denote the known vectors $A\mathbf{u}_k^* - 4\mathbf{u}_{k-1}^* - B\mathbf{u}_{k-2}^*$. Then $A\boldsymbol{\delta}_k = \mathbf{R}_k$ and therefore $\boldsymbol{\delta}_k = A^{-1}\mathbf{R}_k$. Since the eigenvalues of A are known to be larger than 2, it follows that the eigenvalues of A^{-1} are smaller than unity and therefore

$$\| \boldsymbol{\delta}_k \| = \| A^{-1}\mathbf{R}_k \| \leqq \| \mathbf{R}_k \|.$$

We conclude that δ^* in equation (26*) is the largest of the norms of the n "residual vectors" $\mathbf{R}_k = A\mathbf{u}_k^* - 4\mathbf{u}_{k-1}^* - B\mathbf{u}_{k-2}^*$. These vectors will depend, of course, on the specific method of computing the $\mathbf{u}_{k+1}$'s from (6).

A discussion of two alternative schemes for solving implicit systems of equations of the type (6) is contained in [3].

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2. A. N. Lowan, *The Operator Approach to Problems of Stability and Convergence*, Scripta Mathematica, Yeshiva University, New York, 1957, p. 55, p. 82–86.
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Rotors in Polygons and Polyhedra

By Michael Goldberg

1. Introduction. Curves of constant width are closed curves which can be rotated through all orientations between two fixed parallel straight lines while remaining tangent to these lines. They are also known as curves of constant breadth, Gleichdicke (in German), or orbiformes (in French). They have been studied by mathematicians beginning with Euler [1], Minkowski [3], Blaschke [8, 9], Schilling [11], and others to the present time. They have been applied in mechanisms to generate a wide variety of periodic motions. They have been used as the shapes of drills for drilling square and hexagonal holes. They have been produced unintentionally in certain manufacturing processes when only precise circular cylinders were desired.

Two more fixed parallel lines may be added without constraining the rotation. In particular, the four lines may form a square. Hence, the curve may rotate while remaining tangent to all sides of the square. For this reason I call the curve a *rotor in a square*.

Immediately, there arises the question whether non-circular rotors exist for other polygons. It has been found that they exist for all regular polygons and various methods of deriving them have been developed. The earliest complete development was published in 1909 by Meissner, a Swiss mathematician [4]. He derived the rotors in the n-gons and described them analytically by means of the polar tangential equation

$$p = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta)$$

where p is the distance from the origin to the tangent to the curve, θ is the angle which the normal makes with the reference axis, the a_k and b_k are arbitrary constants except that

$$a_k\,, b_k = 0 \quad \text{for} \quad k \neq \pm 1 \pmod{n}.$$

(For example, if $n = 6$, the terms which do not have to be zero are obtained for $k = 1, 5, 7, 11, 13, 17, 19$, etc.) Hence, for each polygon there is an infinity of different rotors. For convexity, the constants a_k and b_k are limited by certain inequality relationships.

2. Circular-Arc Rotors. Special cases have received special attention. In particular, those rotors which are bounded entirely by arcs of circles have been considered. Euler considered the rotor made of three equal arcs, each centered on a vertex of an equilateral triangle. Reuleaux [2] considered rotors of an odd number of circular arcs of equal radii. They have been called Reuleaux polygons, but I prefer to call them Reuleaux rotors since they are not polygons. The regular Reuleaux rotors are rotors in a square.

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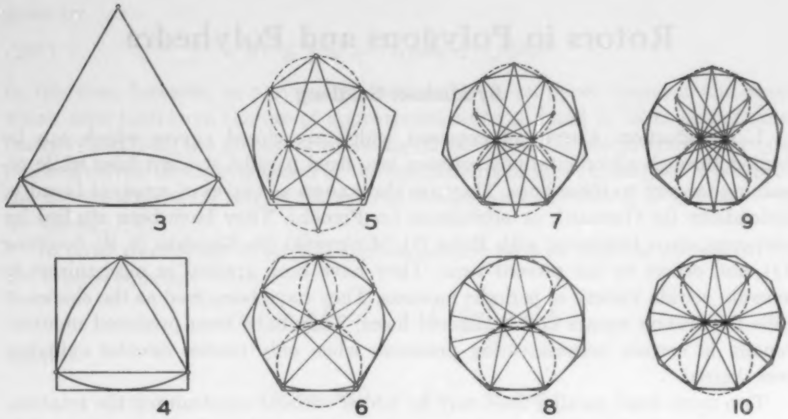

FIG. 1.—Circular-arc rotors in regular polygons.

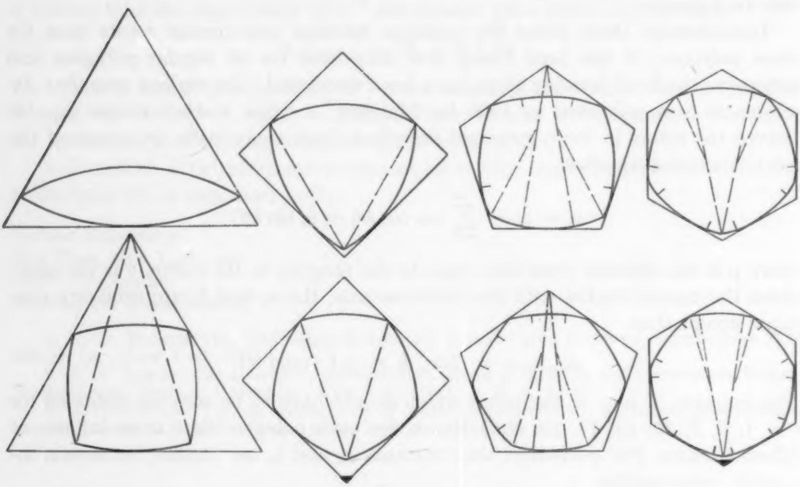

FIG. 2.—Regular trammel rotors in regular polygons.

Fujiwara [14] considered rotors in a triangle. He showed that some of the Reuleaux rotors are also rotors in a triangle. Besides these, he derived the rotors of two, four and five circular arcs. No such rotors of three arcs exist. Of the possible rotors of two arcs, one had already been published by Reuleaux [2]. A circular-arc rotor for the pentagon is also described by Fujiwara [14, pp. 245–246]. In 1948, I published a paper describing the construction of circular-arc rotors for *all* the regular polygons [29]. The method is shown in Figure 1 where the centers of the arcs are regularly distributed on two generating circles, except for $n = 6$ where

three generating circles are used. Note that for $n > 4$, the rotors for the even polygons possess only one axis of symmetry (unequal generating circles), while the rotors in the odd polygons possess two axes of symmetry (equal generating circles). In these, the radii of the arcs are not equal. In particular, note that the rotor in the pentagon is composed of two pairs of arcs, the radii of one pair being twice the radii of the other pair. It is conjectured that, among all the rotors whose contours are made of discrete circular arcs, these have the least number of arcs.

In 1957, I published a new series of circular-arc rotors which are characterized by higher orders of symmetry [32]. They are illustrated in Figure 2. These regular rotors in a regular n-gon are of two types. The upper series is made of $(n - 1)$ equal segments. The lower series is made of $(n + 1)$ equal segments. For $n > 4$, each segment is made of three or more circular arcs, each arc being tangent to its neighboring arcs except at the ends of the segments.

3. Trammel Method of Construction. All the circular-arc rotors may be obtained kinematically by a graphical method which I call a trammel construction. This consists of moving the rotor so that, for each portion of the motion, two points of the rotor trace two fixed straight lines. In Figure 3, the rotor to be generated in the hexagon is based on a regular pentagon shown in dotted lines. The pentagon is turned counter-clockwise so that the vertices B and C move along the straight sides of the hexagon until the next vertex D touches the hexagon. The motion is continued with vertices C and D moving along the sides of the hexagon. During these motions, the sides of the fixed hexagon will mold a rotor based on the dotted pentagon; that is, the rotor is the envelope of the sides of the hexagon on the plane of the moving pentagon.

4. Any Rotor as the Sum of Trammel Rotors. Every rotor in a kn-gon (where k is any integer) is also a rotor in an n-gon. Hence, we have obtained an infinite number of circular-arc rotors for each polygon. Furthermore, if $p_1 = f_1(\theta)$ and $p_2 = f_2(\theta)$ are the polar tangential equations of two trammel rotors in a given polygon, then their weighted mean, given analytically as

$$p_3 = [uf_1(\theta) + vf_2(\theta)]/(u + v)$$

where u and v are any real numbers, is the polar tangential equation of a new rotor in the same polygon. By extension, any rotor in a polygon can be expressed as a

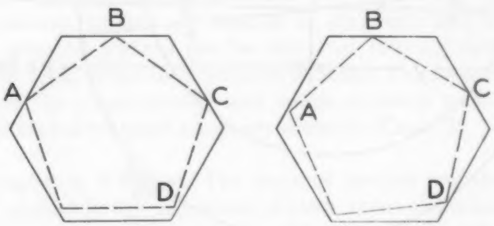

FIG. 3.—Vertices of regular $(n - 1)$-lobed rotor in n-gon.

weighted mean of a series (possibly infinite) of regular trammel rotors. This is similar to the Fourier series expansion used by Meissner as described in Section 1.

5. Basic Rotors and their Construction. In Meissner's equation, each variable term in the right-hand member can serve as the entire variable part of the right-hand member of the equation for a rotor. These rotors, which I call basic rotors, resemble the regular circular-arc rotors [31]. However, the curvature of the arcs varies continuously instead of remaining constant by segments.

Each basic rotor can be obtained kinematically by the following method. Consider a fixed circle of circumference c. Compare with Figure 4. Let another circle of circumference $cn/(n + 1)$, where n is any integer, roll within the first circle without slipping. Then each point of the rolling circle describes a hypocycloid of $(n + 1)$ cusps. Let a straight line be carried by the rolling circle. The envelope of the positions of this straight line will be a parallel curve of the hypocycloid. If the straight line is sufficiently distant from the center of the rolling circle, the envelope will be convex. This convex curve of $(n + 1)$ maximal points or lobes is a basic rotor in a regular n-gon, as shown by the following argument.

When the center of the rolling circle returns to its initial position, the circle and the carried straight line will have undergone a rotation of $2\pi/n$. As the rolling is repeated successively, there will be n symmetric positions of the straight line. Therefore, the n positions of the straight line will form a regular n-gon. The rolling circle can then be considered to carry the regular n-gon. As it rolls, the n envelopes of the n sides are the same. Therefore, each side of the n-gon keeps in contact with the envelope as the n-gon is rotated. Inversely, the $(n + 1)$-lobed rotor can rotate continuously within the n-gon while keeping contact with all the sides of the n-gon.

A similar procedure obtains for $(n - 1)$-lobed rotors in a regular n-gon.

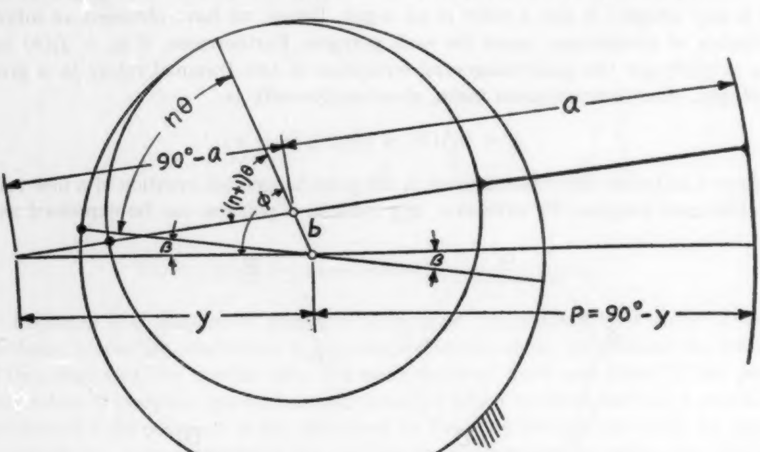

FIG. 4.—Generation of basic rotors.

FIG. 5.—Conical rotors in pyramids.

6. Rotors in Spherical Polygons. The concept of rotors has been generalized from the two dimensions of the plane to three and more dimensions. However, the intermediate case of rotors in spherical polygons has received very little attention. Blaschke [9] and Santaló [25] have considered ovals of constant width on the surface of the sphere. These are ovals which remain tangent to the two meridians which bound a line.

These were generalized by the author to include rotors in all the regular spherical polygons. The trammel rotor method of constructing circular-arc rotors in plane polygons is also applicable for the construction of rotors in spherical polygons. This time, however, the generated arcs are not plane and, therefore, they are not circular [30].

Also, the spherical rotors corresponding to the basic plane rotors can be generated by the rolling circle method. In this case, the rolling circle carries an arc of a great circle instead of a straight line. (See Figure 4.) The symmetric positions of this great circle make a regular spherical polygon [31].

Every rotor in a plane polygon has its counterpart as a rotor in a spherical polygon. However, there are more types of spherical rotors for two reasons. The most obvious reason is that regular spherical polygons of the same number of sides are not similar; their shape depends upon their size. Therefore, their rotors are correspondingly different. But the more surprising difference between plane and spherical rotors is the fact that spherical ovals of constant width, which are tangent to two arcs of great circles, are distinct from rotors in a spherical quadrilateral. In the plane, these two sets of rotors are identical.

Models of spherical surfaces are difficult to construct and to demonstrate. However, every spherical polygon can be converted into a pyramid by passing planes through the great circles by which it is bounded. The rotor in the spherical polygon is replaced by a non-circular cone which is now a rotor in a pyramid. Models of several conical trammel rotors are shown in Figure 5.

7. Rotors Tangent to n Circles. The trammel method and the rolling circle method may be applied in the derivation of other types of rotors. For example, n equal circles may replace the n straight lines of a plane regular n-gon [33]. Among the interesting possibilities is a series of rotors which approximate regular polygons.

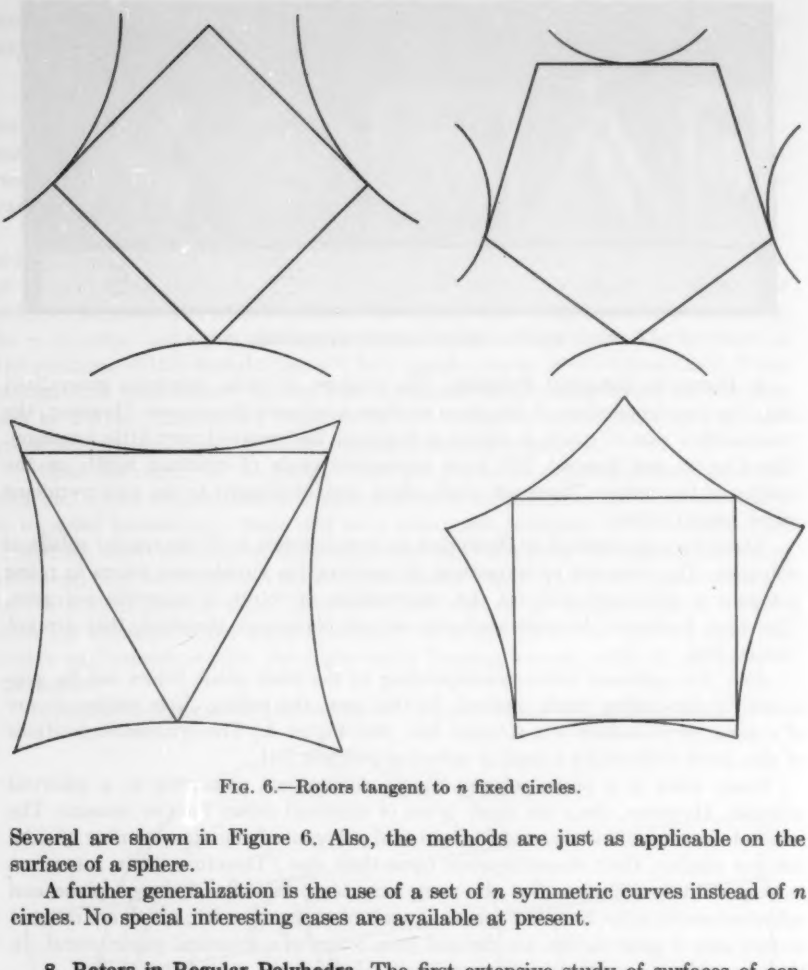

FIG. 6.—Rotors tangent to n fixed circles.

Several are shown in Figure 6. Also, the methods are just as applicable on the surface of a sphere.

A further generalization is the use of a set of n symmetric curves instead of n circles. No special interesting cases are available at present.

8. Rotors in Regular Polyhedra. The first extensive study of surfaces of constant width was made by Minkowski [3]. He devised general geometric methods of deriving many of them. Many beautiful theorems concerning them were obtained. The most obvious method of generating one is by revolving a symmetric oval of constant width about its axis of symmetry. A more unusual surface is based on a regular tetrahedron [6]. Each vertex serves as a center of a spherical surface passing through the other vertices. However, one edge of each pair of opposite edges must be chamfered to a portion of a toroidal surface. The foregoing shapes are shown as A and B of Figure 7.

The surface of constant width is a rotor in a cube. The general investigation of rotors for all the regular polyhedra was first considered by Meissner. In a very

elegant paper [7], he showed that non-spherical rotors exist for the regular tetrahedron, the cube and the regular octahedron, and that they do not exist for the regular dodecahedron (12 faces) and the regular icosahedron (20 faces). Just as Meissner used a Fourier series development for rotors in plane polygons, he used a spherical harmonic development for studying rotors in regular polyhedra. He showed that the rotors in the cube may be described by the polar tangential equation

$$p(\theta, \phi) = a_0 + Y_1 + Y_3 + Y_5 + Y_7 + \cdots ;$$

the rotors in the tetrahedron may be written as

$$p(\theta, \phi) = a_0 + Y_1 + Y_2 + Y_5 ;$$

and the rotors in the octahedron may be written as

$$p(\theta, \phi) = a_0 + Y_1 + Y_5 ,$$

where Y_i (a function of θ and ϕ) is the spherical surface harmonic of the ith degree.

The number of arbitrary parameters in the equation for the rotor in the cube is infinite. The number of parameters for the rotor in the tetrahedron is eleven, while the number of parameters for the rotor in the octahedron is eight. Several rotors for the tetrahedron are shown in Figure 8. A rotor for the octahedron is shown on the extreme right of Figure 7.

Before leaving three dimensions, it may be fitting to consider other possibilities. It is conceivable that non-regular polyhedra may have non-spherical rotors. In particular, it is not known whether open-ended regular prisms can have non-

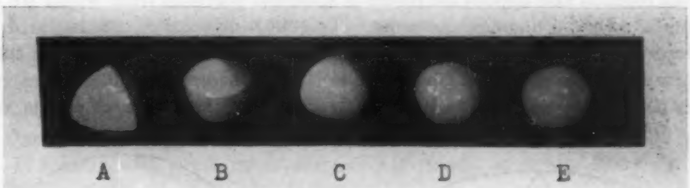

FIG. 7.—Rotors in cube.

A. Rotor based on tetrahedron; B. Rotor of revolution; C. Three-lobed rotor; D. Five-lobed rotor; E. Rotor in octahedron.

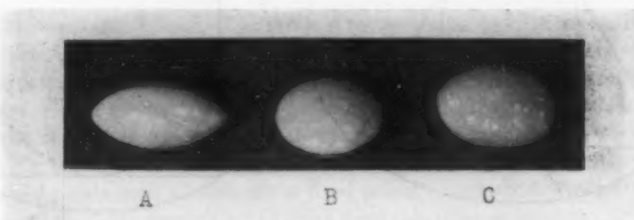

FIG. 8.—Rotors in regular tetrahedron.

A. Prolate rotor of revolution; B. Oblate rotor of revolution; C. Triaxial rotor.

spherical rotors besides the well-known rotors in a rhombic prism which are surfaces of constant width.

9. Rotors in Higher Dimensions. Surfaces of constant width have been generalized to rotors in higher dimensions. Santaló [26] derived relations between measures of their boundaries and their contents as generalizations of relations derived by Minkowski [3, pp. 215–220]. Rotors in the simplex (the analogue of the triangle and the tetrahedron) also exist. Simple examples are the bodies of revolution given by the following polar tangential equations:

$$p = a + b \cos 2\phi, \quad \text{Rotor in simplex,}$$

$$p = a + b \cos 3\phi, \quad \text{Rotor in hypercube.}$$

Non-spherical rotors for the analogue of the octahedron have not been mentioned in the literature.

10. Polygon Rotors in Ovals. If a rotor in a regular polygon is held fixed while the polygon is rotated about it, all the vertices of the polygon trace the same curve. Therefore, if this curve is fixed, the regular polygon can be rotated within it while all the vertices lie on the curve. Hence, for each rotor in a polygon, we have a curve within which the polygon can be rotated. See Figure 9. An application of a triangular rotor in an oval is a recent design, by the German engineer Felix Wankel, of a non-reciprocating internal combustion engine. Test models of this engine have been made for the Curtiss-Wright Corporation.

This relation also applies for rotors in spherical polygons from which we obtain pyramidal rotors in non-circular cones.

However, in three dimensions, this relation does not apply. If a non-spherical rotor in a polyhedron is held fixed while the polyhedron is rotated, the vertices of the polyhedron do not lie on a surface. Instead, the positions of the vertices

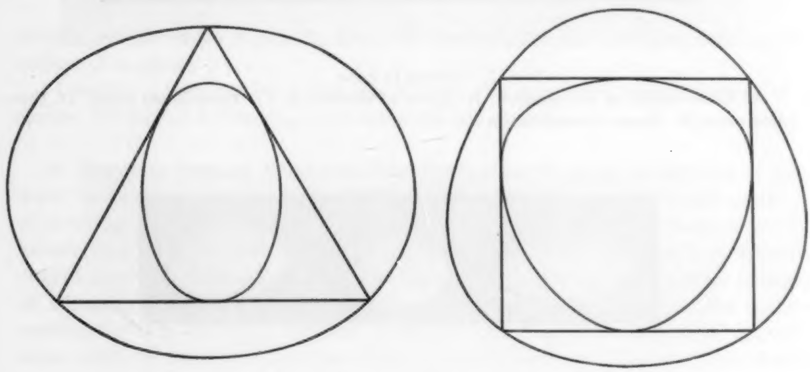

FIG. 9.—(a) Triangle rotor in an oval; (b) Square rotor in an oval.

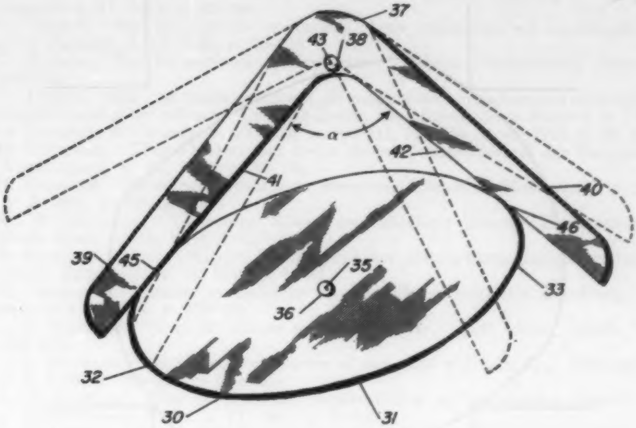

FIG. 10.—Double-contact cam.

fill a volume bounded by two closed surfaces. The ellipsoid is the only known surface possessing the property that the vertices of all the circumscribing rectangular parallelepipeds lie on a sphere. It has been conjectured that every other body generates a volume in this manner and not a surface [10].

11. Mechanisms Related to Rotors. The use of drills in the shape of rotors in polygons has been extended from the drilling of even polygons (the square and the hexagon) to the drilling of odd polygons (the triangle and the pentagon). Several other problems, closely related to rotors in polygons, have been investigated. One is the determination of the non-circular shapes of pivoted rotors which remain in contact with two straight arms of a pivoted rocker arm. See Figure 10. Again, the only admissible angles between the arms are rational fractions of a circle as are the angles of a regular polygon. Examples of the ovals are described by

$$p_1(\theta) = \cos p_2(\theta)$$

where $p_2(\theta) = m\pi/2n + k \sin n\theta$, which is a rotor in a polygon. A geometer would describe these ovals as curves whose isoptic curves are circles [28]. They are the basis of a patent issued for a series of double-contact cam mechanisms [36].

Another related mechanism is the intermittent rotor [34]. This makes contact with a series of fixed elements but not always with all the elements. In the example shown in Figure 11, the rotor is restrained in its motion by contact with three of the four fixed points until all four of the fixed points are touched. The motion may then be continued with another set of three fixed points as constraints.

The presentation of new and unfamiliar basic mechanisms in this paper shows that the ancient science of mechanisms is far from exhausted. When the engineer

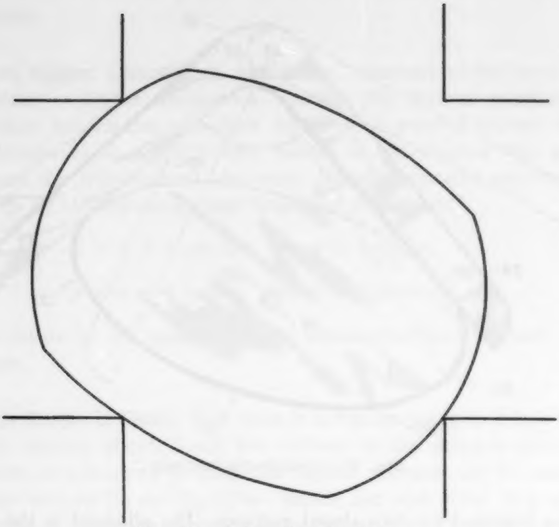

FIG. 11.—Intermittent rotor.

and designer become more familiar with them, it is expected that more applications will be made.

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Numerical Quadrature Over a Rectangular Domain in Two or More Dimensions

Part 3. Quadrature of a Harmonic Integrand

By J. C. P. Miller

1. Introduction. In Part 1 [1], §5, formula (B), §7, formula (B′), and in §9; also in Part 2 [2] in several places, we have seen how the error term is very much reduced if the integrand $f(x, y)$ is a harmonic function, that is, if $\nabla^2 f = 0$. In this note we pursue further this special case, in which especially high accuracy is attainable with few points.

It may not be often that the integrand will have this special form, but it seems worthwhile to develop a few of the interesting formulas. We start by obtaining expansions for n variables, and more extensive ones for two variables, and then obtain and consider special quadrature formulas.

2. Expansions. As in Part 2 [2] §2, we develop $f(x_1, x_2, \cdots, x_n)$ as a Taylor series in even powers of each of the variables x_r. Then, using $\nabla^2 f_0 = 0$ whenever it is applicable, we obtain

$$
(2.1)\quad \left\{
\begin{aligned}
J = I/(2h)^n &= (2h)^{-n}\left(\int_{-h}^{h}\right)^n f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n \\
&= f_0 + \frac{h^4}{5!}\frac{4}{3}\mathfrak{D}^4 f_0 + \frac{h^6}{7!}\frac{16}{3}\mathfrak{J}^6 f_0 + \frac{h^8}{9!}\left(\frac{16}{5}\mathfrak{D}^8 + \frac{192}{5}\mathfrak{Q}^8\right) f_0 \\
&\quad + \frac{h^{10}}{11!}\left(\frac{128}{3}\mathfrak{D}^4\mathfrak{J}^6 + \frac{1280}{3}\mathfrak{P}^{10}\right) f_0 \\
&\quad + \frac{h^{12}}{13!}\left(\frac{64}{7}\mathfrak{D}^{12} + \frac{4096}{35}\mathfrak{J}^{12} + \frac{61184}{105}\mathfrak{D}^4\mathfrak{Q}^8 + \frac{707584}{105}\mathfrak{S}^{12}\right) f_0 \\
&\quad + \cdots
\end{aligned}
\right.
$$

where, as before, extended,

$$
(2.2)\quad \mathfrak{D}^4 f_0 = \sum \frac{\partial^4 f_0}{\partial x_r^2 \partial x_s^2} \qquad \mathfrak{J}^6 f_0 = \sum \frac{\partial^6 f_0}{\partial x_r^2 \partial x_s^2 \partial x_t^2} \qquad \mathfrak{Q}^8 f_0 = \sum \frac{\partial^8 f_0}{\partial x_r^2 \partial x_s^2 \partial x_t^2 \partial x_u^2}
$$

etc., the summations extending over all possible combinations of r, s, t, $\cdots$ with no two equal.

Labelling the symmetrical sets of points as in Part 2, we have likewise the expansions for sums of values of f over the sets

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$$(2.31)\quad 0 \qquad f_0$$

$$(2.32)\quad \alpha(a) \quad 2nf_0 - \frac{4h^4a^4}{4!}\mathfrak{D}^4f_0 + \frac{6h^6a^6}{6!}\mathfrak{Z}^6f_0 + \frac{4h^8a^8}{8!}(\mathfrak{D}^8 - 2\mathfrak{Q}^8)f_0$$
$$- \frac{10h^{10}a^{10}}{10!}(\mathfrak{D}^4\mathfrak{Z}^6 - \mathfrak{P}^{10})f_0$$
$$- \frac{2h^{12}a^{12}}{12!}(2\mathfrak{D}^{12} - 3\mathfrak{Z}^{12}$$
$$- 6\mathfrak{D}^4\mathfrak{Q}^8 + 6\mathfrak{S}^{12})f_0 + \cdots$$

$$(2.33)\quad \beta(b) \quad 2n(n-1)f_0 - \frac{8h^4b^4}{4!}(n-4)\mathfrak{D}^4f_0 + \frac{12h^6b^6}{6!}(n-16)\mathfrak{Z}^6f_0$$
$$+ \frac{8h^8b^8}{8!}\{(n+6)\mathfrak{D}^8 - 2(n-64)\mathfrak{Q}^8\}f_0$$
$$- \frac{20h^{10}b^{10}}{10!}\{(n-4)\mathfrak{D}^4\mathfrak{Z}^6 - (n-256)\mathfrak{P}^{10}\}f_0$$
$$- \frac{4h^{12}b^{12}}{12!}\{2(n-34)\mathfrak{D}^{12} - 3(n+362)\mathfrak{Z}^{12}$$
$$- 6(n-728)\mathfrak{D}^4\mathfrak{Q}^8 + 6(n-1024)\mathfrak{S}^{12}\}f_0 + \cdots$$

$$(2.34)\quad \gamma(c,d) \quad 4n(n-1)f_0 - \frac{8h^4}{4!}\{(n-1)(c^4+d^4) - 6c^2d^2\}\mathfrak{D}^4f_0$$
$$+ \frac{12h^6}{6!}\{(n-1)(c^6+d^6) - 15c^2d^2(c^2+d^2)\}\mathfrak{Z}^6f_0$$
$$+ \frac{8h^8}{8!}[\{(n-1)(c^8+d^8) - 28c^2d^2(c^4+d^4) + 70c^4d^4\}\mathfrak{D}^8$$
$$- 2\{(n-1)(c^8+d^8) - 28c^2d^2(c^4+d^4) - 70c^4d^4\}\mathfrak{Q}^8]f_0$$
$$+ \cdots$$

$$(2.35)\quad \epsilon(e) \quad \frac{4}{3}n(n-1)(n-2)f_0 - \frac{8h^4e^4}{4!}(n-2)(n-7)\mathfrak{D}^4f_0$$
$$+ \frac{12h^6e^6}{6!}(n^2 - 33n + 122)\mathfrak{Z}^6f_0$$
$$+ \frac{8h^8e^8}{8!}\{(n-2)(n+13)\mathfrak{D}^8 - 2(n^2 - 129n + 1094)\mathfrak{Q}^8\}f_0$$
$$+ \cdots$$

We recall that 0 is the origin, or centre of the square, $\alpha(a)$ includes all points with one coordinate $\pm ah$ and the rest zero, $\beta(b)$ has two coordinates each independently $\pm bh$ and the rest zero, $\gamma(c, d)$ has one coordinate $\pm ch$, another $\pm dh$ and the rest zero, and finally $\epsilon(e)$ has three coordinates each independently $\pm eh$ with the rest zero.

3. Expansions over a Square. Such expansions are simpler since $\mathfrak{J}^6 f_0$, $\mathfrak{Q}^8 f_0$ etc., are absent. They can be obtained by analysis with the detached operators—in particular $\mathfrak{D}$; we proceed to obtain expansions with general terms.

If $F(z) = u + iv$ is a function of a complex variable $z = x + iy$ then both u and v are harmonic functions satisfying $D_x^2\phi + D_y^2\phi = 0$. Likewise, if u is a harmonic function, it can be shown that v exists such that $u + iv$ is a function of a complex variable. We then have

$$D_y F = iF' = iD_x F$$

and

$$D_x D_y = \mathfrak{D}^2 = iD_x^2 = -iD_y^2. \tag{3.1}$$

In order to develop expansions we therefore substitute

$$D_x = i^{-1/2}\mathfrak{D} \qquad D_y = i^{1/2}\mathfrak{D}. \tag{3.2}$$

Consider, firstly

$$J = (2h)^{-2}\int_{-h}^{h}\int_{-h}^{h} f(x, y)\,dx\,dy = \frac{1}{4h^2 D_x D_y}(e^{hD_x} - e^{-hD_x})(e^{hD_y} - e^{-hD_y})f_0. \tag{3.3}$$

The operator is

$$\left\{\begin{aligned} \frac{\sinh hD_x \sinh hD_y}{h^2 D_x D_y} &= \frac{\sinh i^{-1/2}h\mathfrak{D} \sinh i^{1/2}h\mathfrak{D}}{h^2\mathfrak{D}^2} \\ &= \frac{1}{2}\,\frac{\cosh (i^{1/2} + i^{-1/2})h\mathfrak{D} - \cosh (i^{1/2} - i^{-1/2})h\mathfrak{D}}{h^2\mathfrak{D}^2} \\ &= \frac{1}{2}\,\frac{\cosh \sqrt{2}h\,\mathfrak{D} - \cos \sqrt{2}h\mathfrak{D}}{h^2\mathfrak{D}^2} \end{aligned}\right. \tag{3.4}$$

whence

$$J = \left(\frac{2}{2!} + \frac{2^3h^4}{6!}\mathfrak{D}^4 + \frac{2^5h^8}{10!}\mathfrak{D}^8 + \cdots + \frac{2^{2r+1}h^{4r}}{(4r+2)!}\mathfrak{D}^{4r} + \cdots\right)f_0. \tag{3.5}$$

Likewise

$$\left\{\begin{aligned} \sum_{\alpha(a)} f(x_\alpha, y_\alpha) &= (e^{ahD_x} + e^{-ahD_x} + e^{ahD_y} + e^{-ahD_y})f_0 \\ &= 2\,(\cosh ahD_x + \cosh ahD_y)f_0 \\ &= 2\,(\cosh i^{-1/2}ah\mathfrak{D} + \cosh i^{1/2}ah\mathfrak{D})f_0 \\ &= 4\cosh\frac{ah}{\sqrt{2}}\mathfrak{D}\cos\frac{ah}{\sqrt{2}}\mathfrak{D}\, f_0 \\ &= 4\left[1 - \frac{a^4h^4}{4!}\mathfrak{D}^4 + \frac{a^8h^8}{8!}\mathfrak{D}^8 + \cdots + (-1)^r\frac{a^{4r}h^{4r}}{(4r)!}\mathfrak{D}^{4r} + \cdots\right]f_0 \end{aligned}\right. \tag{3.6}$$

and

$$\left\{\begin{aligned} \sum_{\beta(b)} f(x_\beta, y_\beta) &= 4\cosh bhD_x \cosh bhD_y\, f_0 \\ &= 4\cosh i^{-1/2}bh\mathfrak{D}\cosh i^{1/2}bh\mathfrak{D}\, f_0 \\ &= 2\,(\cosh bh\sqrt{2}\mathfrak{D} + \cos bh\sqrt{2}\mathfrak{D})f_0 \\ &= 4\left[1 + \frac{2^2b^4h^4}{4!}\mathfrak{D}^4 + \frac{2\,b^8h^8}{8!}\mathfrak{D}^8 + \cdots + \frac{2^{2r}b^{4r}h^{4r}}{(4r)!}\mathfrak{D}^{4r} + \cdots\right]f_0. \end{aligned}\right. \tag{3.7}$$

We shall not use all the expansions given above in the present note, but it seems useful to set out the collected results for future use.

4. Lattice-point Formulas over a Square. We consider first formulas in two variables, and start with 9 points, putting $a = b = 1$ and using the sets 0, $\alpha(1)$, $\beta(1)$. We write

$$J = I/4h^2 = A_0 f(0, 0) + \sum A_\alpha f(x_\alpha, y_\alpha) + \sum A_\beta f(x_\beta, y_\beta) \tag{4.1}$$

using (x_α, y_α) etc. as typical sets of coordinates.

Using (3.5) to (3.7), we equate coefficients of $\mathfrak{D}^{4r} f_0$, $r = 0(1)2$. This gives

$$\begin{cases} A_0 + 4\,A_\alpha + 4\,A_\beta = 1 \\ - 4\,A_\alpha + 16\,A_\beta = \frac{4}{15} \\ + 4\,A_\alpha + 64\,A_\beta = \frac{16}{15} \end{cases} \tag{4.2}$$

with correction term $\qquad C = -\left(-4A_\alpha + 256A_\beta - \dfrac{64}{91}\right)\dfrac{h^{12}}{12!}\mathfrak{D}^{12} f_0.$

We obtain the formula

$$\boxed{\begin{matrix} 7 & -32 & 7 \\ -32 & 1000 & -32 \\ 7 & -32 & 7 \end{matrix}} \div 900 \quad \text{with main correction term } -\frac{1952}{1365}\frac{h^{12}}{12!}\mathfrak{D}^{12} f_0. \tag{4.3}$$

This formula is remarkably good. With the example of Part I, we have, writing $J' = h^2 J$

$$J' = \frac{1}{4}\int_0^{1.2}\int_0^{1.2} \sin x \sinh y \, dx\, dy = \frac{1}{4}(1 - \cos 1.2)(\cosh 1.2 - 1)$$
$$\doteqdot 0.12922\ 70590\ 73675\ 11602$$

Formula (4.3) gives

$$J' \doteqdot 0.12922\ 70590\ 72834\ 11029$$

with $E = -0.0^{12}841\ 00573$ and $C = +0.0^{12}841\ 01633$.

5. Five-point Formulas. The high precision of (4.3) suggests that formulas of lesser precision, with fewer points, may be useful. We use the first two of (4.2) and take one of A_0, A_α, A_β to be zero.

(i) $A_0 = 0$ gives an eight-point formula with relatively poor precision.

$$\boxed{\begin{matrix} 19 & 56 & 19 \\ 56 & 0 & 56 \\ 19 & 56 & 19 \end{matrix}} \div 300 \quad \text{with main correction term } -40\frac{h^8}{9!}\mathfrak{D}^8 f_0. \tag{5.1}$$

(ii) $A_\alpha = 0$ gives

$$\boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 56 & 0 \\ 1 & 0 & 1 \end{matrix}} \div 60 \quad \text{with main correction term } -\frac{32}{5}\frac{h^8}{9!}\mathfrak{D}^8 f_0. \tag{5.2}$$

(iii) $A_\beta = 0$ gives

$$\text{(5.3)} \qquad \boxed{\begin{array}{ccc} 0 & -1 & 0 \\ -1 & 19 & -1 \\ 0 & -1 & 0 \end{array}} \div 15 \qquad \text{with main correction term } + \frac{28}{5}\frac{h^8}{9!}\mathfrak{D}^8 f_0 .$$

We observe that (5.2) and (5.3) combine in the proportions $\frac{7}{15}:\frac{8}{15}$ to give (4.3), though without an error estimate! Likewise $\frac{7}{3} \times (5.2) - \frac{4}{3} \times (5.3)$ gives (B) of Note I, and an estimate for the correction, namely $-\dfrac{112}{5}\dfrac{h^8}{9!}\mathfrak{D}^8 f_0$ when $f(x, y)$ is harmonic.

Another combination, that of (5.2) and (5.3) in equal proportions, gives a small correction term:

$$\text{(5.4)} \qquad \boxed{\begin{array}{ccc} 1 & -4 & 1 \\ -4 & 132 & -4 \\ 1 & -4 & 1 \end{array}} \div 120 \qquad \text{with main correction term } - \frac{4}{5}\frac{h^8}{9!}\mathfrak{D}^8 f_0 .$$

Again $4 \times (5.2) - 3 \times (5.3)$ gives small multipliers:

$$\text{(5.5)} \qquad \boxed{\begin{array}{ccc} 1 & 3 & 1 \\ 3 & -1 & 3 \\ 1 & 3 & 1 \end{array}} \div 15 \qquad \text{with main correction term } - \frac{212}{5}\frac{h^8}{9!}\mathfrak{D}^8 f_0 .$$

Evidently (4.3) is most precise, but simultaneous use of (5.2) and (5.3) gives an idea of the precision attained, and readily yields the better result if desired. Formula (5.5) might be helpful with desk computing, but (5.1) has little to recommend it.

Numerical results for some of the formulas using the example of §4 are as follows:

Formula	Result J'	$10^{10} \times E$	$10^{10} \times C$
(5.1)	0.12922 72986	+2395	−2396
(5.2)	0.12922 70974	+383	−383
(5.3)	0.12922 70255	−336	+335
I(B)	0.12922 71932	+1341	−1342
(5.4)	0.12922 70615	+24	−24

6. General n; $2n^2 + 1$ Points. We consider now the n-dimensional case, $n \geqq 3$, using lattice points 0, $\alpha(1)$, $\beta(1)$. In this case the term in $\mathfrak{J}^6 f_0$ is relevant, and the $\mathfrak{Q}^8 f_0$ term will appear in the error, except when $n = 3$.

We equate coefficients of f_0, $\mathfrak{D}^4 f_0$, $\mathfrak{J}^6 f_0$ in the expansions resulting from use of (2.1), (2.31)–(2.33) in (4.1). We obtain

$$(6.1)\qquad \begin{cases} A_0 + 2n\,A_\alpha + 2n(n-1)\;A_\beta = 1 \\ \quad -\;4\,A_\alpha - \;8(n-4)\;A_\beta = \frac{4}{15} \\ \quad +\;6\,A_\sigma + 12(n-16)\;A_\beta = \frac{16}{21} \end{cases}$$

while

$$C = -\left\{ 4A_\alpha + 8(n+6)A_\beta - \frac{16}{45} \right\} \frac{h^8}{8!}\,\mathfrak{D}^8 f_0$$

$$+ \left\{ 8A_\alpha + 16(n-64)A_\beta + \frac{192}{45} \right\} \frac{h^8}{8!}\,\mathfrak{C}^8 f_0 .$$

These yield

$$(6.2)\qquad A_0 = \frac{-61n^2 + 931n + 3780}{3780} \qquad A_\alpha = \frac{61n - 496}{3780} \qquad A_\beta = -\frac{61}{7560}$$

with

$$C = \frac{1198}{945}\frac{h^8}{8!}\,\mathfrak{D}^8 f_0 + \frac{3619}{315}\frac{h^8}{8!}\,\mathfrak{C}^8 f_0 .$$

In particular

$$(6.33)\qquad n = 3 \qquad A_0 = \frac{12048}{7560} \qquad A_\alpha = -\frac{626}{7560} \qquad A_\beta = -\frac{61}{7560}$$

$$(6.34)\qquad n = 4 \qquad A_0 = \frac{13056}{7560} \qquad A_\alpha = -\frac{504}{7560} \qquad A_\beta = -\frac{61}{7560}$$

$$(6.35)\qquad n = 5 \qquad A_0 = \frac{13820}{7560} \qquad A_\alpha = -\frac{382}{7560} \qquad A_\beta = -\frac{61}{7560}$$

$$(6.36)\qquad n = 6 \qquad A_0 = \frac{14340}{7560} \qquad A_\alpha = -\frac{260}{7560} \qquad A_\beta = -\frac{61}{7560} .$$

As a numerical illustration for $n = 3$ we consider

$$J = \frac{1}{8} I = \frac{1}{8} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \cos\frac{3}{4}x \cos y \cosh\frac{5}{4}z\,dx\,dy\,dz$$

$$= \frac{16}{15} \sin\frac{3}{4} \sin 1 \sinh\frac{5}{4} = 0.9800827.$$

Formula (6.33) gives $J \doteq 0.9799734$ with $E = -0.0000109$ and $C = +0.0000110$.

This result is less spectacular than that of §4, for these reasons:

i) In §4, $h = 0.6$, here $h = 1$, and the correction term in (4.3) contains a high power of h.

ii) The correction term in (6.2) is of order h^8, that in (4.3) is of order h^{12}

iii) The higher the number of dimensions, the more individual terms there are in $\mathfrak{D}^8 f$, $\mathfrak{D}^{12} f$, etc. In (4.3) there is only one term in $\mathfrak{D}^{12} f$, in (6.33) there are 9 in $\mathfrak{D}^8 f$.

iv) The effect of larger interval h is enhanced by the use of the factor $\frac{5}{4}$, which exceeds unity, in $\cosh\frac{5}{4}z$; this is only partially balanced by the factor $\cos\frac{3}{4}x$.

In spite of these points, the formula (6.2) seems a good one.

7. Quadrature over a Square; Specially Chosen Points. Since the expansions

of §3 involve only cross-differences $\mathfrak{D}^4 f_0$, it appears likely that use of sets of diagonal points β will be more profitable than attempts to use sets α. It turns out that sets 0, $\alpha(a)$, $\beta(b)$ and 0, $\alpha(a)$ both faii to give real values of α if maximum precision is sought. On the other hand, we can get several formulas making use of any number of sets $\beta(b_p)$, $p = 0(1)r$, both with and without the point 0.

We start first with r sets $\beta(b_p)$, without the point 0. We have to find the $2r$ constants A_{β_p}, b_p satisfying the equations

$$\sum_{p=1}^{r} 4A_{\beta_p} b_p^{4(s-1)} = \frac{1}{(2s-1)(4s-3)} = C_{s-1}, \qquad s = 1(1)2r \tag{7.1}$$

obtained by substitution of (3.5) to (3.7) in

$$J = \sum A_{\beta_p} f(\pm b_p h, \pm b_p h) \tag{7.2}$$

and equating the coefficients of the first $2r$ coefficients of $\mathfrak{D}^4$. Sundry powers of 4 have been cancelled.

By familiar arguments, the b_p^4 are roots of the equation

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^r \\ C_0 & C_1 & C_2 & \cdots & C_r \\ C_1 & C_2 & C_3 & \cdots & C_{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_r & C_{r-1} & C_{r-2} & \cdots & C_{2r-1} \end{vmatrix} = 0. \tag{7.3}$$

These are the orthogonal polynomials for the weight function $w(x) = \frac{1}{2}(x^{-3/4} - x^{-1/2})$ and range $0 \leqq x \leqq 1$. The first two are

$$\begin{cases} 15x - 1 = 0 \\ 819x^2 - 438x + 11 = 0. \end{cases} \tag{7.4}$$

The main correction term is obtained from the next power of $\mathfrak{D}^4$ and yields

$$C = \left(C_{2r} - \sum_{p=1}^{r} 4A_{\beta_p} b_p^{8r}\right) \frac{2^{4r} h^{8r} \mathfrak{D}^{8r}}{(8r)!} f_0. \tag{7.5}$$

If the point 0 is included, our equations (7.1) are replaced by

$$\begin{cases} A_0 + 4 \sum_{p=1}^{r} A_{\beta_p} = 1 \\ \sum_{p=1}^{r} 4A_{\beta_p} b_p^{4s} = \dfrac{1}{(2s+1)(4s+1)} = C_s, \qquad s = 1(1)2r \end{cases} \tag{7.6}$$

and the b_p^4 are roots of the equation

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^r \\ C_1 & C_2 & C_3 & \cdots & C_{r+1} \\ C_2 & C_3 & C_4 & \cdots & C_{r+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{r+1} & C_{r+2} & C_{r+3} & \cdots & C_{2r} \end{vmatrix} = 0$$

which are the orthogonal polynomials for the weight function $w(x) = \frac{4}{3}(x^{1/4} - x^{1/2})$ for the range $0 \leqq x \leqq 1$. The first two are

$$(7.8) \qquad \begin{cases} 3x - 1 = 0 \\ 17017x^2 - 13650x + 1745 = 0. \end{cases}$$

The main correction term is this time

$$(7.9) \qquad C = \left(C_{2r+1} - \sum_{p=1}^{r} 4A_{\beta_p} b_p^{8r+4}\right) \frac{2^{4r+2} h^{8r+4}}{(8r+4)!} \mathfrak{D}^{8r+4} f_0 .$$

In each case the coefficients A_{β_r} may be computed by standard methods.

8. Formulas for $r = 1$. These have 4 and 5 points respectively

$$(8.1) \qquad A_\beta = \frac{1}{4} \qquad b = 15^{-1/4} \qquad C = \frac{64}{225} \frac{h^8 \mathfrak{D}^8}{8!} f_0$$

$$(8.2) \qquad A_0 = \frac{4}{5} \qquad A_\beta = \frac{1}{20} \qquad b = 3^{-1/4} \qquad C = \frac{2816}{12285} \frac{h^{12} \mathfrak{D}^{12}}{12!} f_0 .$$

Written out in full:

$$(8.3) \quad J = \tfrac{1}{4}I = \tfrac{1}{4}\{f(15^{-1/4}, 15^{-1/4}) + f(-15^{-1/4}, 15^{-1/4}) + f(15^{-1/4}, -15^{-1/4}) + f(-15^{-1/4}, -15^{-1/4})\}$$

$$(8.4) \quad J = \tfrac{1}{4}I = \tfrac{4}{5}f(0, 0) + \tfrac{1}{20}\{f(3^{-1/4}, 3^{-1/4}) + f(-3^{-1/4}, 3^{-1/4}) + f(3^{-1/4}, -3^{-1/4}) + f(-3^{-1/4}, -3^{1/4})\}.$$

As a numerical test use

$$J = \frac{1}{4} I = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \cos x \cosh y \, dx \, dy = \sin 1 \sinh 1 = 0.98889\ 77057\ 62865.$$

Formula (8.3) gives 0.98889 06525 with

$$E = -0.00000\ 70533 \text{ and } C = +0.00000\ 70547$$

and formula (8.4) gives 0.98889 77062 41358 with

$$E = +0.0^9 4\ 78493 \text{ and } C = -0.0^9 4\ 78543.$$

9. Formulas for $r = 2$. These have 8 and 9 points respectively

$$(9.1) \qquad \begin{array}{ll} b_1 = 0.40316\ 26030\ 59346\ 89754 & A_{\beta_1} = 0.22912\ 30654\ 28169\ 97222 \\ b_2 = 0.84439\ 75319\ 23478\ 74713 & A_{\beta_2} = 0.02087\ 69345\ 71830\ 02778 \end{array}$$

with main correction term $\frac{54592}{57014685} \times \frac{256}{16!} h^{16} \mathfrak{D}^{16} f_0$

$$(9.2) \qquad \begin{array}{ll} b_0 = 0 & A_0 = 0.69521\ 80834\ 12925\ 81989 \\ b_1 = 0.63205\ 02078\ 18796\ 99524 & A_{\beta_1} = 0.06686\ 42185\ 46105\ 38162 \\ b_2 = 0.89531\ 63791\ 24106\ 97730 & A_{\beta_2} = 0.00993\ 12606\ 00663\ 16340 \end{array}$$

with main correction term $\dfrac{16832}{78975897} \times \dfrac{1024}{20!} h^{20}\mathfrak{D}^{20}f_0$.

For

$$J = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \cos x \cosh y \, dx \, dy$$

formula (9.1) gives 0.98889 77057 62853 38396 with

$$E = -0.0^{13}\,1171243 \quad \text{and} \quad C = +0.0^{13}1171555$$

while formula (9.2) gives 0.98889 77057 62865 09647 with

$$E = +0.0^{19}9 \quad \text{and} \quad C = -0.0^{19}90.$$

With formula (9.2) we find approximately 0.82447 37090 77903 16756 for

$$\frac{1}{16} \int_{-2}^{2} \int_{-2}^{2} \cos x \cosh y \, dx \, dy = \sin 2 \sinh 2 \doteqdot 0.82447\,37090\,77809\,15433$$

with $E = +0.0^{13}9401323$ and $C = -0.0^{13}9406250$.

These formulae clearly have high precision, even with considerable values of h.

10. Quadrature over a Cube; Specially Chosen Points. The search for such formulas is more difficult in 3 or more dimensions. It seems that one or more extra available constants are needed in order to obtain real points. We shall not pursue this, but give one simple formula for three dimensions.

We find nothing convenient by use of points $\alpha(a)$, with or without 0; likewise 0 with $\beta(b)$ fails to give real points. We can, however, use 12 points $\beta(b)$ alone. We have then to satisfy

$$(10.1) \qquad \begin{cases} 2n(n-1)A_\beta = 1 \\ 8b^4(4-n)A_\beta = \frac{4}{15}, \qquad \text{where } n = 3. \end{cases}$$

This yields $\quad b = (2/5)^{1/4} = 0.79527\ 07287\ 67051 \qquad A_\beta = 1/12$

with main correction term

$$C = \left(156\, b^6 A_\beta + \frac{16}{21}\right) \frac{h^6}{6!} \mathfrak{Z}^6 f_0 = 0.005626 h^6 \mathfrak{Z}^6 f_0.$$

With the example of §6, with integrand $\cos \frac{3}{4}x \cos y \cosh \frac{5}{4}z$ (10.1) gives $J \doteqdot 0.97519$ with $E = -0.00489$ and $C = +0.00494$.

The only formula found that allows for the term $\mathfrak{Z}^6 f_0$ and has an error of order h^8 is (6.33), which needs 19 points. It is evident that further search is needed.

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A Method for Calculating Solutions of Parabolic Equations with a Free Boundary

By Milton E. Rose

1. Introduction. The description of phenomena involving phase transitions leads to a class of parabolic partial differential equations with a free interior boundary which marks the interface separating the phases. The simplest nonlinear problem of this type was first treated by J. Stefan in 1891 and considerable attention has been focused on such problems in recent years [4], [6], [9].

A typical mathematical formulation of this type of problem is the following:

PROBLEM I: Determine functions $x(t)$ and $u(x, t)$ satisfying

$$(1) \qquad \begin{aligned} &(a)\ u_{xx}(x, t) = u_t(x, t), && 0 < x < x(t), \quad t > 0 \\ &(b)\ u_x(0, t) = -1, && t > 0 \\ &(c)\ \dot{x}(t) = -u_x(x(t), t), && t > 0 \\ &(d)\ u(x, t) = 0, && x \geqq x(t), \quad t \geqq 0 \\ &(e)\ x(0) = 0. \end{aligned}$$

With suitable choice of units this formulation describes, for example, the recrystallization due to uniform heating at one end of a semi-infinite substance which is initially at its critical temperature [3]. Equations (1c), (1d), and (1e) serve to describe the interface separating the phases of crystalline state. Effective numerical procedures and proofs of their validity have been described by Douglas and Gallie [2], and Trench [8].

The object of this paper is to suggest a computational approach to general problems of this type which has the feature that the path of the interface is not regarded as an explicitly imposed interior boundary condition. The method is closely related to a proposal by P. Lax [7] for calculating weak solutions of hyperbolic equations in conservation form as the limits of suitable finite difference equations. Several problems involving shocks and contact discontinuities have been successfully treated by Lax from this point of view. The present investigation was motivated by the question of whether or not phase-change interfaces were susceptible to a similar treatment, particularly in view of the fact that such interfaces also arise as discontinuity surfaces in the general treatment of the equations of hydrodynamics (Keller [5]).

In the first part of this paper a formulation of certain phase transition phenomena is described (Problem II) and the concept of their weak solutions is introduced. Heuristic arguments are given serving to identify a weak solution of Problem II with the solution of Problem I. The results of several calculations are cited to lend support to a conjecture that the solution of certain natural finite difference equa-

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tions related to Problem II converge to a relevant (essentially unique) weak solution of the problem. In addition, a random walk process suggested by the difference equation is mentioned which tends to illuminate certain aspects of phase transition processes and suggests an additional computational approach to such problems.

The author wishes to express his appreciation to P. Lax and E. Isaacson for their interest and discussions concerning this paper and to N. Metropolis for discussions of the random walk formulation.

2. Weak Solutions of Phase Transition Processes. We begin by deriving equations governing a particular example of a process involving a phase transition.* Consider [3] a bar of uniform cross-section of a substance which undergoes a change in crystalline form with negligible change in density involving a latent heat of crystallization. Let e denote the specific internal energy, q the heat flux, T the temperature and ρ the density. The equation

$$(\rho e)_t + q_x = 0 \tag{2}$$

expresses the conservation of energy of the process (we use the notation $v_x = \partial v/\partial x$, etc.).

We assume Fourier's law,

$$q = -kT_x \tag{3}$$

where k is the coefficient of thermal conductivity. Finally recalling our assumption that the density is constant, we assume an equation of state given by

$$T = T(e). \tag{4}$$

In a region in which this transformation is one to one we obtain

$$T_t = \alpha^2 T_{xx}$$

with $\alpha^2 = k/\rho e_T$. This is the usual equation of heat conduction. However for processes in which a latent heat of recrystallization occurs the transformation (4) is no longer one to one; the following relationship is typical of such processes (see Figure 1).

$$T = \gamma(e) + T_0 \tag{4'}$$

with

$$\gamma(e) = \begin{cases} \gamma^- \cdot (e - e_0), & e < e_0 \\ 0, & e_0 \leqq e \leqq e_0 + H \\ \gamma^+ \cdot [e - (H + e_0)], & e > e_0 + H. \end{cases}$$

Here H is the latent heat of recrystallization and γ^- and γ^+ are non-negative quantities related to the specific heat of the material. Without loss of generality we may assume $e_0 = 0$ and $T_0 = 0$.

If we adjoin to equations (2), (3) and (4′) appropriate initial and boundary conditions it seems reasonable to suppose our formulation describes a unique and well-posed physical problem. Summarizing, we formulate

* Our discussion is easily generalized to several space dimensions.

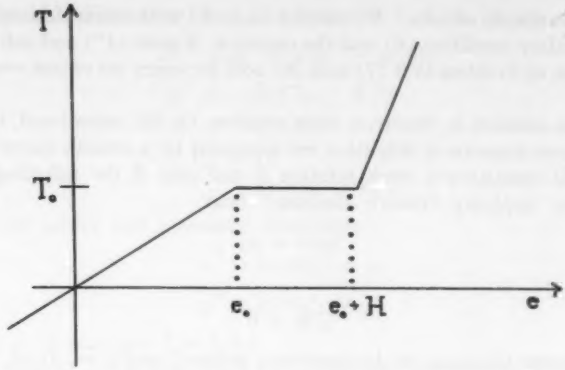

FIG. 1.—Equation of state for a recrystallization process.

PROBLEM II: Find solutions $e(x, t)$, $q(x, t)$ and $T(x, t)$ of the equations

$$(\rho e)_t + q_x = 0, \qquad x > 0,\ t > 0, \tag{2}$$

$$q + kT_x = 0, \qquad x > 0,\ t > 0, \tag{3}$$

$$T = \begin{cases} \gamma^- e, & e < 0 \\ 0, & 0 \leqq e \leqq H \\ \gamma^+(e - H), & e > H \end{cases} \tag{4''}$$

satisfying the initial and boundary conditions

$$e(x, 0) = e_0(x), \qquad x > 0 \tag{5}$$

$$q(0, t) = g(t), \qquad t > 0 \tag{6}$$

Because of the mild nonlinearity introduced by equation (4″) our formulation of Problem II may not allow a genuine solution, i.e., a solution which is continuous with continuous derivatives in the quarter plane D $[x > 0, t > 0]$ for a compact set of initial and boundary data. The physical situation suggests that the class of solutions be enlarged to allow jump discontinuities along certain smooth curves C in D.

We do this by introducing in the usual manner the concept of a weak solution of Problem II. Consider a space Φ of pairs of smooth testing functions (φ, ψ) with compact support, i.e., each (φ, ψ) in Φ vanishes identically outside some compact bounded region which lies in the half space $t > 0$ and does not intersect the boundary $x = 0$. We multiply equation (2) by φ and equation (3) by ψ, integrate and then integrate by parts to obtain

$$\iint (\varphi_t e + \varphi_x q)\, dx\, dt + \int \varphi(x, 0) e_0(x)\, dx = 0 \tag{7}$$

and

$$\iint (q\psi - kT\psi_x)\, dx\, dt = 0 \tag{8}$$

recalling that $e(x, 0) = e_0(x)$. We restrict (e, q, T) to the class of functions satisfying the boundary condition (6) and the equation of state (4″) and call (e, q, T) a weak solution of Problem II if (7) and (8) hold for every set of test vectors (φ, ψ) in Φ.

A genuine solution is, clearly, a weak solution. On the other hand, two genuine solutions whose domains of definition are separated by a smooth curve C given by $x = x(t)$ will constitute a weak solution if and only if the following equations (obtained by applying Green's theorem) hold:

$$[\rho e]\dot{x} = [q]$$

and

$$[T]\dot{x} = 0$$

where $[v]$ denotes the jump in the quantity v across C and $\dot{x} =_s dx/dt$. Using (3), (4″) and the assumption that ρ is constant we obtain

$$\rho H\dot{x} = -[kT_x] \tag{9}$$

and

$$[T] = 0 \tag{10}$$

which express the conditions of balance of heat flux and the continuity of temperature across the interface. These conditions correspond to the conditions expressed in equations (1c) and (1d) of Problem I; they result in Problem II as a requirement for a weak solution.

I conjecture that a weak solution of Problem II exists which is composed of piecewise genuine solutions; furthermore it is the only such solution which is continuously dependent on the initial conditions with respect to some appropriate norm. If this conjecture can be proved correct, the identification of this solution with the solution of the Stefan problem, Problem I, would follow directly through proper choice of the initial and boundary conditions. Probably the solution of Problem II can be obtained as the limit of a certain set of finite difference equations which are described below. More precisely,

Conjecture A: There is a unique weak solution (e, q, T) of Problem II composed of genuine solutions in each of the regions separated by the interface curve $x = x(t)$; furthermore, the solution u of Problem I and the solution (e, q, T) of Problem II (with suitable initial and boundary conditions) as identified through the relationship $T(e) = u$ are identical.

Conjecture B: (e, q, T) may be obtained as the strong limit of the sequence of solutions of the finite difference equations described below (Problem II_Δ).

I wish to give certain plausibility arguments which, together with the results of certain calculations described later, tend to support these conjectures. In doing so I lean heavily on the already cited paper by Lax [7].

The following describes the simplest finite difference approximant to Problem II: at a point (i, n), $i = 1, 2, \cdots$, $n = 1, 2, \cdots$, of a mesh Δ of size $(\Delta x, \Delta t)$ replace e_t by $(e_i^{n+1} - e_i^n)/\Delta t$, q_x by $(q_i^n - q_{i-1}^n)/\Delta x$, and T_x by $(T_{i+1}^n - T_i^n)/\Delta x$. This leads to

PROBLEM II_Δ : Find solutions e_i^n, q_i^n, and T_i^n of the equations

$$(2_\Delta) \qquad e_i^{n+1} = e_i^n - \mu(q_i^n - q_{i-1}^n)$$

$$(3_\Delta) \qquad q_i^n = -\bar{k}(T_{i+1}^n - T_i^n)$$

$$(4''_\Delta) \qquad T_i^n = \begin{cases} \gamma^- e_i^n, & e_i^n < 0 \\ 0, & 0 \leqq e_i^n \leqq H \\ \gamma^+(e_i^n - H), & e_i^n > H \end{cases}$$

satisfying the initial and boundary conditions

$$(5_\Delta) \qquad e_i^0 = e_0(i\Delta x) \equiv e_{0i}$$

$$(6_\Delta) \qquad q_0^n = g(n\Delta t) \equiv g^n$$

Here, $\mu = \Delta t/\rho\Delta x$ and $\bar{k} = k/\Delta x$.

Let us suppose initial and boundary conditions for which $e_{0i} \geqq 0$ and $g^n \geqq 0$. It is not difficult to show then that $e_i^n \geqq 0$ and the above equations can be simplified. Letting

$$p_i^n = \begin{cases} 0 & \text{if} \quad 0 \leqq e_i^n \leqq H \\ \gamma^+\mu\bar{k} & \text{if} \quad e_i^n > H \end{cases}$$

and eliminating q_i^n and T_i^n in (2_Δ), (3_Δ) we obtain

$$(11) \qquad (e_i^{n+1} - H) = p_{i+1}^n(e_{i+1}^n - H) + p_{i-1}^n(e_{i-1}^n - H) + (1 - 2\,p_i^n)(e_i^n - H)$$

for $n \geqq 0$, $i \geqq 2$, and

$$(12) \qquad (e_i^{n+1} - H) = p_2^n(e_2^n - H) + (1 - p_1^n)(e_1^n - H) + \mu\bar{k}g^n$$

for $n \geqq 0$.

Very likely the stability and convergence of this system is insured by requiring that the coefficients $(1 - 2\,p_i^n)$ be non-negative. This leads to the condition

$$(13) \qquad \frac{\Delta t}{\Delta x^2} \leqq \frac{\rho}{2k\gamma^+}$$

From the definition of p_i^n it is easily seen from (11) that, if e_{i+1}^n, e_i^n, and e_{i-1}^n are each in the interval $[0, H]$, then $e_i^{n+1} = e_i^n$. Suppose, for example, that initially $e_{0i} = 0$. We may conclude that *the width of the transition region between phases, i.e., the region in which e_i^n varies from its initial value* 0 *to a value greater than H, is $2\Delta x$ for any line t_n = constant.* This property provides a striking contrast to the spreading of shock zones in Lax's treatment of hydrodynamic problems and suggests that phase transitions can be determined accurately by this method.

It is of interest to consider the interpretation of equation (11) as a random walk process. Consider wells each capable of holding H balls at points P_0, P_1, $\cdots$ situated on a line. A ball in a well which is not yet filled is required to remain in the well ($p_i^n = 0$); a ball at a point at which the well is already filled moves to the right or left with equal probability $p_i^n > 0$ or remains at this point with probability $(1 - 2\,p_i^n)$.

The boundary condition q_0^n describes the rate at which balls enter the process

at the left and the capacity of each well measures the latent heat of the process. In addition to their theoretical interest, random walk processes of this type suggest feasible computational methods for more general, but related, problems.

3. Numerical Computations. In this section the results of several calculations based on the difference scheme outlined in Problem II are described for the following particular problem:

$$e_t + q_x = 0$$

$$q + T_x = 0$$

$$T = \begin{cases} 0, & 0 \leq e \leq 1 \\ \frac{1}{2}e, & e < 0 \end{cases}$$

In each case the initial and boundary data were

$$e(x, 0) = 1, \qquad x > 0$$

and

$$q(0, t) = -1, \qquad t > 0.$$

The interface curve $x(t)$ was determined as the distance from the origin to the first meshpoint on the line t_n = constant for which $T = 0$. For comparison (c.f. eq. (9))

$$\frac{dx^*}{dt} = T_x(x(t), t)$$

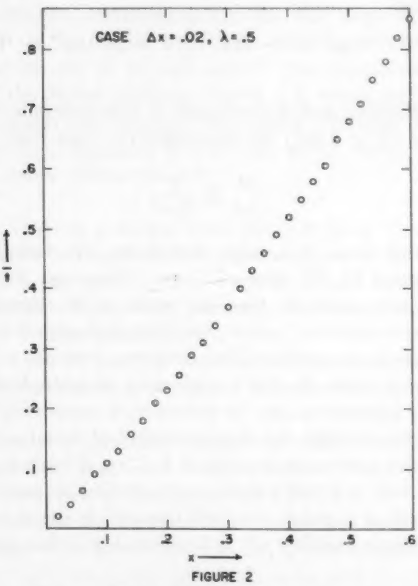

FIG. 2.—Interface for $\Delta x = .02$, $\lambda = .5$.

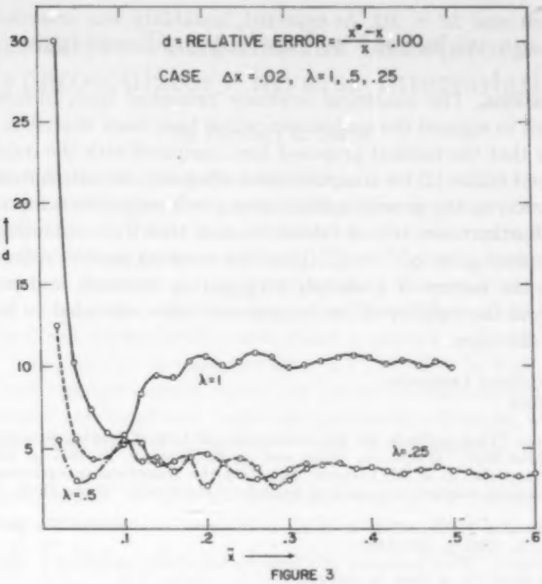

Fig. 3.—Dependence of relative error on λ.

TABLE 1

Comparison of values $t(x, \Delta x)$, $(\Delta t/\Delta x^2 = \frac{1}{2})$

x	$t(x, .1)$	$t(x, .02)$
0.1	0.110	0.108
0.2	0.225	0.232
0.3	0.365	0.370
0.4	0.515	0.519
0.5	0.675	0.679
0.6	0.845	—*
0.7	1.025	—*
0.8	1.220	—*

* Values not calculated.

was integrated numerically using a backward difference for T_x at $x(t)$ and the percent relative error

$$d = 100\left[\frac{x^*(t) - x(t)}{x(t)}\right]$$

calculated.

Figure 2 shows the interface for the case where $\Delta x = .02$ and $\Delta t/\Delta x^2 = .5$. Comparison to the case in which $\Delta x = .1$ and $\Delta t/\Delta x^2 = .5$ may be made from Table 1 which compares $t(x; \Delta x)$ for these two cases.

In Figure 3 the dependence of the relative error d on the value $\lambda = \Delta t/\Delta x^2$ is

given. In each case $\Delta x = .02$. As expected, instability was observed for values of $\lambda > 1$, although this fact has not been indicated in the figure.

4. Conclusions. The numerical evidence presented here, although somewhat limited, seems to support the conjectures which have been described in this paper. It is unlikely that the method proposed here compares with the implicit technique of Douglas and Gallie [2] for computational efficiency, at least for one-dimensional problems. However, the present method seems well suited for computation in more dimensions. Furthermore, it is of interest to note that if, in obtaining Problem II_Δ, we were to replace q_x by $[q_i^{n+1} - q_{i-1}^{n+1}]/\Delta x$, the resulting implicit difference equations would retain the feature of a sharply propagating interface and raise additional questions about the validity of our conjectures when extended to include implicit equations of this type.

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Alternative Formulas for Osculatory and Hyperosculatory Inverse Interpolation

By Herbert E. Salzer

1. Introduction. In previous papers devoted to inverse osculatory interpolation, all of which were based upon the employment of the Lagrange-Hermite formulas for direct interpolation of $f(x)$ (at equal intervals h for real functions or at points of a Cartesian grid of length h for complex functions) including both the osculatory (function with first derivative) and hyperosculatory (function with first and second derivative) types, the writer has given the Taylor series obtained by the inversion of the direct interpolation series about a suitable point x_0, in powers of $r = [f(x) - f(x_0)]/hf'(x_0)$ through the term in r^{10} [1]–[3].* In fact, the same procedure has been employed much earlier for inverse interpolation formulas based upon the ordinary non-osculatory Lagrange interpolation formula, which gives the inversion series in powers of a variable r that is proportional to $f(x) - f(x_0)$ [4]–[6].*

In this present note we give an alternative scheme for inverse osculatory or hyperosculatory interpolation in terms of the inverse function $x(f)$, involving $x(f)$ and $dx(f)/df$, or $x(f)$, $dx(f)/df$ and $d^2x(f)/df^2$ at separate points $f_i \equiv f(x_i)$. In other words, now we distinguish between the previously given point expansions in [1]–[3], which may be characterized as special types of "inverse osculatory or hyperosculatory" formulas and the present "osculatory or hyperosculatory inverse" formulas, which are analogous in structure to the osculatory or hyperosculatory direct interpolation formulas.

2. Advantages of Alternative Scheme.

A. The present scheme avoids any cumbersome explicit formulas by applying both a decomposability and uniqueness property of the Lagrange-Hermite interpolation formula, which has been so effective in reducing the labor in direct interpolation of high degree (described in detail in [3] and [7]), to those same kinds of osculatory formulas for the inverse functions.

B. The formulas here, in terms of $x(f)$ with either $x'(f)$ or $x'(f)$ and $x''(f)$ at $f = f_i$, enable the user to go far beyond the 10th degree in accuracy with a fraction of the computational labor required for the power series formulas. Actual count of the number of operations for 10th degree accuracy by the older method and 11th degree accuracy (either 6-point osculatory or 4-point hyperosculatory) using the present scheme, showed the latter to involve only around one-fourth of the number of operations required in the former.

C. These alternative formulas are more truly interpolatory because of the actual agreement with the inverse function and its derivatives at different points over the complete range, and the consequent closeness of the approximation near a

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* This article is intended to be self-sufficient in the presentation of these alternative schemes for inverse interpolation. We shall avoid as much as possible the repetition of material in [1]–[8] to which the reader is referred for full details.

number of points; whereas the convergence of the Taylor series expansion becomes rapidly poorer as $|r|$ exceeds 1.*

D. In this alternative scheme, since the f_i's are the fixed arguments and are unequally spaced, it hardly matters whether the corresponding x_i's are equally or unequally spaced. But for real variables, the previously given formulas in [1] and [3] would not be applicable when the x_i's are irregularly spaced.

E. For complex osculatory or hyperosculatory inverse interpolation for $z = z(f)$ when $f(z)$ is an analytic function tabulated in the complex plane with $f'(z)$, or with $f'(z)$ and $f''(z)$, there is no change in these present formulas other than the replacement of x by z.

3. Osculatory and Hyperosculatory Inverse Interpolation Formulas. Before giving the alternative scheme for the osculatory and hyperosculatory cases, we should mention for the sake of completeness that even ordinary inverse interpolation has a corresponding alternative form. Instead of the formulas in [4]–[6], we may prefer the following concise rearrangement of Lagrange's interpolation polynomial for the inverse function $x(f)$:

$$x = \sum \alpha_i x_i / \sum \alpha_i, \tag{1}$$

where

$$\alpha_i \equiv A_i/(f - f_i), \tag{2'}$$

and

$$A_i \equiv 1/\prod_{j \neq i} (f_i - f_j). \tag{2''}$$

In the above (1), (2′), (2″), as well as in all subsequent formulas, x denotes either real or complex values, and in any $\sum$ or $\prod$ the running index i, j or k has whatever range is customary, say $-[(n-1)/2]$ to $[n/2]$ for real interpolation or over any set of fixed grid points in complex interpolation (except for omissions indicated beneath the symbol).

For all osculatory and hyperosculatory formulas we employ the first and second derivatives of the inverse function $x(f)$, namely $x'(f) = 1/f'(x)$ and $x''(f) = -f''(x)/[f'(x)]^3$, at $x = x_i$, the same as at $f(x) = f(x_i)$, or more concisely, at $f = f_i$. The notation x_i' and x_i'' is used for $x'(f_i)$ and $x''(f_i)$ respectively. It is not necessary to repeat here for the inverse function the development given in [3], [7] and [8] for concise expressions for the direct osculatory and hyperosculatory interpolation formulas, since those same ideas apply here.

For the inverse function, in the osculatory and hyperosculatory formulas we

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* To see why, say in the real case, in a region where $x(f)$ is one-valued and has derivatives of high enough order, consider the remainder term in the n-point osculatory or hyperosculatory Lagrange-Hermite formula for $x(f)$, namely

$$(1/(2n)!)\{\textstyle\prod_{i=-[(n-1)/2]}^{[n/2]} (f - f_i)\}^2 (d^{2n}x/(df)^{2n}) \,|_{f=f_\xi}$$

or

$$(1/(3n)!)\{\textstyle\prod_{i=-[(n-1)/2]}^{[n/2]} (f - f_i)\}^3 (d^{3n}x/(df)^{3n}) \,|_{f=f_\xi}$$

which can become very small for an f very close to f_i, even though far from f_0.

need the first and second derivatives of $L_i^{(n)}(f)$, the n-point Lagrange interpolation coefficients in f defined by

$$(3) \qquad L_i^{(n)}(f) \equiv \prod_{j \neq i} (f - f_j) / \prod_{j \neq i} (f_i - f_j).$$

Differentiation is with respect to f, after which we set $f = f_i$.* These derivatives are conveniently expressed as follows:

$$(4') \qquad \frac{d}{df} L_i^{(n)}(f) \mid_{f=f_i} = \sum_{j \neq i} 1/(f_i - f_j),$$

which is written more concisely, employing the notation

$$(5) \qquad 1/(f_i - f_j) \equiv \alpha_{ij},$$

as

$$(4) \qquad L_i^{(n)\prime}(f_i) = \sum_{j \neq i} \alpha_{ij}.$$

Differentiating $L_i^{(n)}(f)$ twice and setting $f = f_i$, we obtain from (3),

$$(6'') \qquad \left.\frac{d^2 L_i^{(n)}(f)}{df^2}\right|_{f=f_i} = 2 \sum_{j \neq i} \sum_{\substack{k \neq i \\ \text{and } j \neq k}} 1/(f_i - f_j)(f_i - f_k),$$

the outside factor of 2 occurring because in every (j, k) combination there will be a (k, j) combination, $j \neq k$. The double summation occurring in (6″) is avoided by employing the identity $2 \sum\sum \alpha_{ij}\alpha_{ik} = (\sum \alpha_{ij})^2 - \sum \alpha_{ij}^2$ so that

$$(6') \qquad L_i^{(n)\prime\prime}(f_i) = (\sum_{j \neq i} \alpha_{ij})^2 - \sum_{j \neq i} \alpha_{ij}^2,$$

which from (4) is simply

$$(6) \qquad L_i^{(n)\prime\prime}(f_i) = \{L_i^{(n)\prime}(f_i)\}^2 - \sum_{j \neq i} \alpha_{ij}^2.$$

For osculatory interpolation, following [7] p. 213, we define

$$(7') \qquad a_i = A_i^2$$

which from (2″) and (5) may be expressed as

$$(7) \qquad a_i = \{\prod_{j \neq i} \alpha_{ij}\}^2,$$

and

$$(8') \qquad b_i = -2A_i^2 L_i^{(n)\prime}(f_i),$$

which from (2″), (4) and (5) is expressible as

$$(8) \qquad b_i = -2 \{\prod_{j \neq i} \alpha_{ij}\}^2 \sum_{j \neq i} \alpha_{ij}.$$

From (7) and (8) we define

$$(9) \qquad \alpha_i = a_i/(f - f_i)^2 + b_i/(f - f_i),$$

---

* For the occurrence of derivatives of Lagrangian coefficients in direct interpolation see [7] p. 213, for the osculatory case, and [3] p. 105, for the hyperosculatory case.

and

$$\beta_i = a_i/(f - f_i). \tag{10}$$

Finally for the n points x_i where we have both $f(x_i)$ and $f'(x_i)$, we find for x an approximation by the polynomial of the $(2n - 1)$-th degree in $f \equiv f(x)$, which is equal to x_i at $f = f_i$ and whose derivative with respect to f is equal to $x_i' \equiv x'(f_i)$ at $f = f_i$, according to

$$x \sim \sum (\alpha_i x_i + \beta_i x_i')/\sum \alpha_i. \tag{11}$$

For hyperosculatory interpolation, following [3] p. 105, and taking into account (2″), (4) and (5), we define

$$a_i = \{\prod_{j\neq i} \alpha_{ij}\}^3, \tag{12}$$

$$b_i = -3\,\{\prod_{j\neq i} \alpha_{ij}\}^3 \sum_{j\neq i} \alpha_{ij}, \tag{13}$$

and a quantity c_i which in standard notation is given by

$$c_i = A_i^3[-\tfrac{3}{2}L_i^{(n)\prime\prime}(f_i) + 6\{L_i^{(n)\prime}(f_i)\}^2], \tag{14'}$$

but when taking into account (2″), (4), (5) and (6), is expressible in present notation by

$$c_i = \{\prod_{j\neq i} \alpha_{ij}\}^3[\tfrac{9}{2}\,\{\sum_{j\neq i} \alpha_{ij}\}^2 + \tfrac{3}{2} \sum_{j\neq i} \alpha_{ij}^2]. \tag{14}$$

Next, using (12)–(14), we define

$$\alpha_i = a_i/(f - f_i)^3 + b_i/(f - f_i)^2 + c_i/(f - f_i), \tag{15}$$

$$\beta_i = a_i/(f - f_i)^2 + b_i/(f - f_i), \tag{16}$$

and

$$\gamma_i = a_i/2(f - f_i). \tag{17}$$

Then for the n points x_i where we are given $f(x_i)$, $f'(x_i)$ and $f''(x_i)$, we find for x an approximation by the polynomial of the $(3n - 1)$-th degree in f which, together with its first and second derivative with respect to f, is equal to x_i, x_i' and $x_i'' \equiv -f''(x_i)/[f'(x_i)]^3$ at $f = f_i$, according to

$$x \sim \sum (\alpha_i x_i + \beta_i x_i' + \gamma_i x_i'')/\sum \alpha_i. \tag{18}$$

In using (11) or (18), when there are many inverse interpolations with all the values of f being close to each other so that we have the same fixed points f_i, the quantities a_i and b_i in (7) and (8), or a_i, b_i and c_i in (12)–(14) have to be computed just once, to be used repeatedly in (9)–(11) or (15)–(18).

4. Application to Mathematical Table-Making. In a mathematical table where the inverse function will often be wanted, we might avoid the need for a separate table of the inverse function by supplementing the usual aids to direct interpolation (columns of differences or derivatives) with three, five, or even seven extra columns to facilitate osculatory or hyperosculatory inverse interpolation. For example, we might add three columns of just $x_i' \equiv 1/f'(x_i)$ with a_i and b_i defined by (7)

and (8) to aid just osculatory inverse interpolation by (9)–(11), or five columns of x_i', $x_i'' \equiv -f''(x_i)/[f'(x_i)]^3$, a_i, b_i and c_i defined by (12)–(14) to help in just hyperosculatory inverse interpolation in (15)–(18), or even seven columns of x_i', x_i'', one set of functions a_i, b_i defined by (7), (8) and another set of functions a_i, b_i, c_i, defined by (12)–(14), giving the user a choice of either osculatory or hyperosculatory inverse interpolation.

The use of these supplementary columns would not be restricted to tables of functions for just regularly spaced arguments x_i. Thus we may tabulate these auxiliary quantities a_i, b_i and c_i for tables having real arguments x_i irregularly spaced or for tables having complex arguments in a Cartesian or polar grid. Even for osculatory or hyperosculatory *direct* interpolation in a table whose arguments are irregularly spaced points x_i, real or complex, if given $f_i' \equiv f'(x_i)$, or f_i' and $f_i'' \equiv f''(x_i)$, we may tabulate a_i, b_i by (7), (8), or a_i, b_i, c_i by (12)–(14), and use (9)–(11) or (15)–(18), merely interchanging throughout in (5), (7)–(18) the variables x_i with f_i and x with f.

In the functions a_i, b_i and c_i, the choice of the number n of fixed points should be, where feasible, sufficient to ensure full accuracy in the use of (11) or (18), which cannot be finer than the tabular uncertainty error of around $\epsilon/f'(x_i)$, the ϵ being the error in the value of $f(x_i)$.

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Permanents of Incidence Matrices

By Paul J. Nikolai

1. Introduction. This paper describes the evaluation of the permanents of certain incidence matrices using the UNIVAC Scientific 1103A Computer.

The *permanent* of an n by n matrix $A = [a_{ij}]$ with elements in a commutative ring is defined by the relation

$$\text{per}(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where the sum extends over all $n!$ permutations of $1, 2, \cdots, n$. The permanent is thus similar in definition to the determinant and suggests a theory for permanents analogous to that for determinants. No such theory is available, however, since permanents do not obey the analogue of the basic multiplicative law of determinants

$$\det(AB) = \det(A)\det(B).$$

The analogue of the Laplace expansion is easily shown to remain valid for the permanent, but for computational purposes it is of little help. The calculations described in this paper were made practicable by a computational device given by H. J. Ryser. I shall state Ryser's result, which has heretofore not been published, as a theorem.

THEOREM 1. Denote by A_r a matrix obtained from A by replacing r of the column vectors of A by zero vectors. Denote by $S(A_r)$ the product of the row sums of A_r. Then

$$\text{(1.1)} \quad \text{per}(A) = S(A) - \sum_{C_1^n} S(A_1) + \sum_{C_2^n} S(A_2) - \cdots + (-1)^{n-1} \sum_{C_{n-1}^n} S(A_{n-1})$$

where each sum extends over all the C_r^n ways of forming A_r.

Proof. The proof is based on the Principle of Inclusion and Exclusion or Sieve of Sylvester [2].

Consider an element of the form

$$a = a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where the j_i may assume any of the values $1, 2, \cdots, n$. Let r denote the number of these integers not among $j_1, j_2, \cdots, j_n$. Suppose $r > 0$. Then a appears in (1.1)

$$1 \text{ time in } S(A),$$

$$C_1^r \text{ times in } \sum S(A_1),$$

$$C_2^r \text{ times in } \sum S(A_2),$$

$$\cdots$$

$$C_r^r \text{ times in } \sum S(A_r)$$

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or

$$1 - C_1^r + C_2^r - \cdots + (-1)^r C_r^r = 0$$

times in all.

On the other hand, if $j_1, j_2, \cdots, j_n$ is a permutation of $1, 2, \cdots, n$, $r = 0$ and a appears in (1.1) exactly once.

One obvious advantage in using Theorem 1 is that the number of summands required to calculate per (A) is reduced from $n!$ to 2^n. Other features of this device are noted in Section 5.

2. The v, k, λ Problem. Let it be required to arrange v elements into v sets such that every set contains exactly k distinct elements and such that every pair of sets has exactly λ elements in common, $0 < \lambda < k < v$. This problem is referred to as the v, k, λ *problem* and the resulting arrangement is called a v, k, λ *configuration* or *symmetric balanced incomplete block design.* For a v, k, λ configuration list the elements $X_1, X_2, \cdots, X_v$ in a row and the sets $T_1, T_2, \cdots, T_v$ in a column. Insert 1 in row i and column j if X_j belongs to set T_i and 0 otherwise. In this way is obtained a v by v matrix A of 0's and 1's called the *incidence matrix* of the v, k, λ configuration. It is not difficult to show that

$$\lambda = k(k - 1)/(v - 1)$$

and that

$$|\det (A)| = k(k - \lambda)^{\frac{v-1}{2}}.$$

Two v, k, λ configurations D_1 and D_2 will be termed *isomorphic* if there is a one-to-one correspondence $X_1 \leftrightarrow X_2 = (X_1)\alpha$ between the elements $\{X_1\}$ of D_1 and the elements $\{X_2\}$ of D_2 and a one-to-one correspondence $T_1 \leftrightarrow T_2 = (T_1)\beta$ between the sets $\{T_1\}$ of D_1 and the sets $\{T_2\}$ of D_2, such that if $X_1 \in T_1$, then $(X_1)\alpha \in (T_1)\beta$. An isomorphism of a design with itself is called a *collineation.*

These combinatorial designs and their associated incidence matrices have been extensively studied. An excellent summary of the problem together with an extensive bibliography can be found in [3].

3. Permanents of Incidence Matrices. A set $R = \{X_1, X_2, \cdots, X_n\}$ will be called a *system of distinct representatives* for the subsets $T_1, T_2, \cdots, T_n$ of the set D in case $X_i \in T_i$ and $X_i \neq X_j$ for $i \neq j$, $i = 1, 2, \cdots, n$. The permanent of an incidence matrix possesses combinatorial significance in that it equals the number of systems of distinct representatives of the class of sets $T_1, T_2, \cdots, T_v$. As pointed out in Section 2, the determinant of the incidence matrix of a v, k, λ configuration is an elementary function of v, k, and λ alone. It seemed of interest to know whether or not the permanent possesses a similar property. Before trying calculations on UNIVAC Scientific, it seemed clear that per (A) could not be a simple function of v, k, and λ but it remained an open question whether or not non-isomorphic designs having the same parameters v, k, and λ could possess unequal permanents. The answers to these questions would be of greater interest in the case of finite projective planes with $n + 1$ points per line which are v, k, λ designs with

$v = n^2 + n + 1$, $k = n + 1$, and $\lambda = 1$. Unfortunately, the first case of non-isomorphic planes arises for $n = 9$ with incidence matrices of order 91. Present theory does not permit calculation of the permanent of an incidence matrix of this size. There are known to be no instances of non-isomorphic v, k, λ configurations for $v < 15$. Different 15, 7, 3 designs, however, do exist, and the permanents of their incidence matrices were easily calculated on UNIVAC Scientific.

Nandi [1] has constructed all 15, 7, 3 designs. There are five non-isomorphic examples. Study of these five designs revealed that exactly two which Nandi denotes by $(\gamma\gamma')$ and $(\alpha_1\alpha_1')_1$ possess a collineation of order 7 fixing one element (and one set) and permuting the remaining 14 elements (sets) in two cycles of length 7. For both designs let X; $A_1, A_2, \cdots, A_7$; $B_1, B_2, \cdots, B_7$ denote the 15 elements. Each design has a set $\{A_1, A_2, \cdots, A_7\}$ fixed by the collineation (X) $(A_1A_2 \cdots A_7)$ $(B_1B_2 \cdots B_7)$. $(\gamma\gamma')$ and $(\alpha_1\alpha_1')_1$ can be displayed in the form

$$
\begin{array}{lcl}
\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\} & & \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\} \\
\{X, A_1, A_2, A_4, B_1, B_2, B_4\} & & \{X, A_1, A_2, A_4, B_1, B_3, B_4\} \\
\{X, A_2, A_3, A_5, B_2, B_3, B_5\} & & \{X, A_2, A_3, A_5, B_2, B_4, B_5\} \\
\cdots & & \cdots \\
\{X, A_7, A_1, A_3, B_7, B_1, B_3\} & \text{and} & \{X, A_7, A_1, A_3, B_7, B_2, B_3\} \\
\{A_1, A_2, A_4, B_3, B_5, B_6, B_7\} & & \{A_1, A_2, A_4, B_2, B_5, B_6, B_7\} \\
\{A_2, A_3, A_5, B_4, B_6, B_7, B_1\} & & \{A_2, A_3, A_5, B_3, B_6, B_7, B_1\} \\
\cdots & & \cdots \\
\{A_7, A_1, A_3, B_2, B_4, B_5, B_6\} & & \{A_7, A_1, A_3, B_1, B_4, B_5, B_6\}
\end{array}
$$

respectively. Thus their corresponding incidence matrices appear as follows reflecting the collineation:

$$
\begin{array}{ccccccccccccccc}
0&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&0&1&0&0&0&1&1&0&1&0&0&0\\
1&0&1&1&0&1&0&0&0&1&1&0&1&0&0\\
1&0&0&1&1&0&1&0&0&0&1&1&0&1&0\\
1&0&0&0&1&1&0&1&0&0&0&1&1&0&1\\
1&1&0&0&0&1&1&0&1&0&0&0&1&1&0\\
1&0&1&0&0&0&1&1&0&1&0&0&0&1&1\\
1&1&0&1&0&0&0&1&1&0&1&0&0&0&1\\
0&1&1&0&1&0&0&0&0&0&1&0&1&1&1\\
0&0&1&1&0&1&0&0&1&0&0&1&0&1&1\\
0&0&0&1&1&0&1&0&1&1&0&0&1&0&1\\
0&0&0&0&1&1&0&1&1&1&1&0&0&1&0\\
0&1&0&0&0&1&1&0&0&1&1&1&0&0&1\\
0&0&1&0&0&0&1&1&1&0&1&1&1&0&0\\
0&1&0&1&0&0&0&1&0&1&0&1&1&1&0
\end{array}
\qquad
\begin{array}{ccccccccccccccc}
0&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&1&0&1&0&0&0&1&0&1&1&0&0&0\\
1&0&1&1&0&1&0&0&0&1&0&1&1&0&0\\
1&0&0&1&1&0&1&0&0&0&1&0&1&1&0\\
1&0&0&0&1&1&0&1&0&0&0&1&0&1&1\\
1&1&0&0&0&1&1&0&1&0&0&0&1&0&1\\
1&0&1&0&0&0&1&1&1&1&0&0&0&1&0\\
1&1&0&1&0&0&0&1&0&1&1&0&0&0&1\\
0&1&1&0&1&0&0&0&0&1&0&0&1&1&1\\
0&0&1&1&0&1&0&0&1&0&1&0&0&1&1\\
0&0&0&1&1&0&1&0&1&1&0&1&0&0&1\\
0&0&0&0&1&1&0&1&1&1&1&0&1&0&0\\
0&1&0&0&0&1&1&0&0&1&1&1&0&1&0\\
0&0&1&0&0&0&1&1&0&0&1&1&1&0&1\\
0&1&0&1&0&0&0&1&1&0&0&1&1&1&0
\end{array}
$$

The permanent of each matrix is the sum of the equal minors belonging to the fixed row which represents the fixed set. This row and one column containing a 1 in this row can be deleted reducing the order of the matrix by one. The permanent of the reduced matrix is computed and multiplied by the number of 1's per row to yield the permanent of the incidence matrix.

4. Results. The computational advantage offered by the collineations made the choice of $(\gamma\gamma')$ and $(\alpha_1\alpha_1')_1$ for a first trial a natural one. The permanents of

the incidence matrices of these designs were found to be 24, 601, 472 and 24, 567, 424 respectively. Thus nonisomorphic designs having the same parameters v, k, and λ may have unequal permanents, so that the permanent is not a function of v, k, and λ alone.

The program for evaluation of these permanents turned out to be an efficient one requiring about four minutes of computer time for each matrix. The remaining 15, 7, 3 designs were also run with the following results:

Design	Permanent
$(\alpha_1\alpha_1')_2$	24,572,288
$(\alpha_2\alpha_2')$	24,567,424
$(\beta_1\beta_1')$	24,582,016

As computer time became available it was decided to calculate the permanents of the incidence matrices of all cyclic designs with prime v and $v < 23$. Cyclic designs with their simple structure and with v a prime might have yielded possible clues to a formula for the permanent in the cyclic case or to possible divisibility properties. Unfortunately no distinct cyclic designs with the same parameters arise for $v < 31$. Results of this survey are given as follows:

v	k	λ	per (A)
7	3	1	24
7	4	2	144
11	5	2	12,105
11	6	3	75,510
13	4	1	3,852
13	9	6	64,803,969
19	9	4	142,408,674,153
19	10	5	952,709,388,762

In addition, the permanent of the incidence matrix of the projective plane of order 4, a cyclic 21, 5, 1 design, was computed and found to be 18, 534, 400. Here, as with two of the 15, 7, 3 designs, the use of a collineation reduced computation time by more than one half. This was a significant saving, better than four hours of computer time.

5. Description of the Code for the UNIVAC Scientific Computer. Ones complement binary arithmetic, two address logic, and an extensive array of logical instructions together with equation (1.1) applied to 0,1 matrices contributed to a short, fast computer code for UNIVAC Scientific. The v rows of the square matrix A of 0's and 1's were stored in the higher order v stages of v consecutive storage cells. An index r, $0 \leqq r \leqq 2^v - 1$, counted the number of A_r's formed, and served as a logical multiplier in forming A_r. If no row of A_r were zero, $S(A_r)$ was calculated and added or subtracted from the accumulated sum according as the number of 1's in the binary representation of r was even or odd. A_{r-1} was formed next and the calculation continued. The magnitude of the final sum was then per (A).

All loops of the code were checked using the v by v matrix S of all 1's. per $(S) = v!$. Machine accuracy was checked by running each calculation twice in every case.

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sota. Calculations were completed at Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. The author expresses his thanks to Dr. Ernest T. Parker for suggesting the problem, to Remington Rand UNIVAC and to the Digital Computation Branch at Wright Air Development Center for providing time on their UNIVAC Scientific computers.

Wright Air Development Center
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On the Numerical Treatment of Heat Conduction Problems with Mixed Boundary Conditions

By Arnold N. Lowan

Abstract. The two-dimensional problem of heat conduction in a rectangle where the temperature is prescribed over a portion of the boundary while the temperature gradient is prescribed over the remainder of the boundary, may be treated numerically by replacing the differential equation of heat conduction and the equations expressing the given initial and boundary conditions by their difference analogs

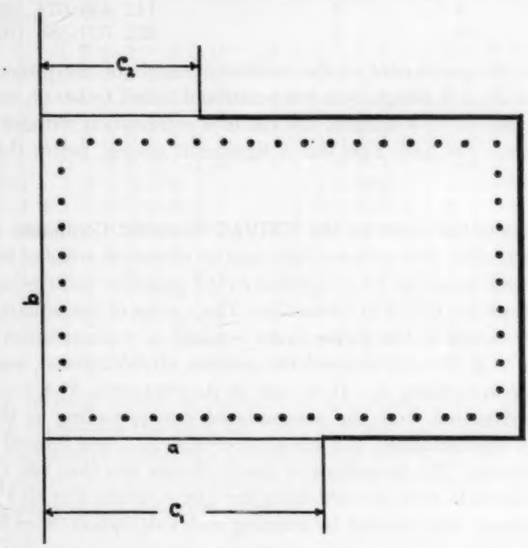

Fig. 1.—Rectangular domain with "mixed" boundary conditions.

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and solving the resulting system. It is shown that if the scheme is to be stable the intervals Δx and Δy must be chosen so that $k\Delta t/(\Delta x)^2 + k\Delta t/(\Delta y)^2 \leqq \frac{1}{2}$.

Consider the two-dimensional problem of heat conduction in a rectangle, Figure 1, when the temperature is prescribed over the thin line portion of the boundary, while the temperature gradient is prescribed over the heavy line portion of the boundary. This is a typical problem with "mixed" boundary conditions and should not be confused with the considerably simpler problem when the temperature is prescribed over certain complete sides of the rectangle, while the temperature gradient is prescribed over the remaining sides. As far as the writer is aware no analytical solution of the mixed boundary value problem above formulated (or of the analogous problem for the cylinder) is to be found in the literature. We must therefore (if interested in numerical answers) resort to the alternative of substituting for the differential equation of heat conduction and for the equations expressing the initial and boundary conditions their appropriate difference analogs, and solving the resulting system.

The mathematical formulation of the problem is as follows:

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad 0 \leqq x \leqq a,\ 0 \leqq y \leqq b,\ t > 0 \tag{1}$$

$$T(x, y, 0) = f(x, y) \tag{2}$$

$$T(0, y, t) = 0 \qquad 0 \leqq y \leqq b \tag{3}$$

$$T(x, 0, t) = 0 \qquad 0 \leqq x \leqq c_1 \tag{4}$$

$$\left[\frac{\partial}{\partial y} T(x, y, t)\right]_{y=0} = 0 \qquad c_1 \leqq x \leqq a \tag{5}$$

$$\left[\frac{\partial}{\partial x} T(x, y, t)\right]_{x=a} = 0 \qquad 0 \leqq y \leqq b \tag{6}$$

$$\left[\frac{\partial}{\partial y} T(x, y, t)\right]_{y=b} = 0 \qquad c_2 \leqq x \leqq a \tag{7}$$

$$T(x, b, t) = 0 \qquad 0 \leqq x \leqq c_2 \tag{8}$$

where for the sake of simplicity we have at first assumed that the prescribed temperature and temperature gradient are $= 0$. The difference analogs of the above equations are:

$$\begin{aligned} T_{h,k,n+1} = \beta\, T_{h,k-1,n} + \alpha\, T_{h-1,k,n} + (1 - 2\alpha - 2\beta)\, T_{h,k,n} \\ + \alpha\, T_{h+1,k,n} + \beta\, T_{h,k+1,n} \qquad h = 1, 2, 3, \cdots M,\ k = 1, 2, 3 \cdots N \end{aligned} \tag{1*}$$

$$T_{h,k,0} = f(h\Delta x, k\Delta y) \tag{2*}$$

$$T_{0,k,n} = 0 \qquad 1 \leqq k \leqq N \tag{3*}$$

$$T_{h,0,n} = 0 \qquad 1 < h < c_1/\Delta x \tag{4*}$$

$$T_{h,1,n} = T_{h,0,n} \qquad c_1/\Delta x \leqq h \leqq a/\Delta x \tag{5*}$$

$$(6^*) \qquad T_{M+1,k,n} = T_{M,k,n} \qquad 1 \leqq k \leqq N$$

$$(7^*) \qquad T_{h,N+1,n} = T_{h,N,n} \qquad c_2/\Delta x \leqq h \leqq a/\Delta x$$

$$(8^*) \qquad T_{h,N+1,n} = 0 \qquad 1 \leqq h \leqq c_2/\Delta x$$

where

$$T_{h,k,n} = T(h\Delta x, k\Delta y, n\Delta t); \qquad \alpha = \frac{k\Delta t}{(\Delta x)^2}; \qquad \beta = \frac{k\Delta t}{(\Delta y)^2}$$

$$\Delta x = a/(M+1) \quad \text{and} \quad \Delta y = b/(N+1).$$

It will be convenient to consider the MN temperatures $T_{h,k,n}$ with $h = 1, 2, 3, \cdots M$ and $k = 1, 2, 3, \cdots N$ as the components of an $M \times N$ − dimensional vector to be denoted by $\mathbf{T}_n$. It will also be convenient to replace the two subscripts h and k identifying the lattice point P_{hk} (i.e., the point with coordinate $x = h\Delta x$ and $y = k\Delta y$) by the single subscript p running from $p = 1$ to $p = MN$ with the understanding that for the M lattice points corresponding to $k = 1$, p runs from 1 to M, for the next set of M lattice points corresponding to $k = 2$ p runs from $M + 1$ to $2M, \cdots$ etc. The system of MN equation (1*) may then be written in the matrix—vector form

$$(9) \qquad \mathbf{T}_{n+1} = A\mathbf{T}_n .$$

As is well known, to prove the stability of (9), it suffices to prove that if S denotes the largest of the sums of absolute values of the elements of the rows of A then $S \leqq 1$.

Let Ω denote the set of lattice points closest to the boundary. It is clear that if the difference equation (1*) is applied to lattice points lying inside of Ω, the resulting equation (which is in fact equation (1*) itself) has the five non-vanishing coefficients β, α, $1 - 2\alpha - 2\beta$, α and β. If we assume $1 - 2\alpha - 2\beta \geqq 0$ or $\alpha + \beta \leqq \frac{1}{2}$ it is clear that the sum of the absolute values of the coefficients is $= 1$. We shall show that if (1*) is applied to lattice points belonging to the set Ω, the resulting equation may be characterized by the fact that the sum of absolute values of the coefficients is smaller than unity. Consider for instance the form taken by (1*) when applied to a point of Ω such that $h\Delta x \leqq c_1$. Since for such a point $T_{h,0} = 0$ the resulting equation has the four non-vanishing coefficients α, $1 - 2\alpha - 2\beta$, α and β. If again we assume $1 - 2\alpha - 2\beta > 0$, it follows that the sum of the absolute values of the coefficients is $= 1 - \beta < 1$. In an entirely similar manner it is shown that if (1*) is applied to lattice points in Ω for which $h\Delta x \geqq c_2$ and the boundary condition (5*) is taken into account, the sum of the absolute values of the coefficients of the resulting equation is $1 - \alpha < 1$. Similar conclusions may be drawn in the case of all lattice points in Ω. Thus the quantity S previously defined is $= 1$. The stability of the difference scheme under consideration is thus proven, provided that the intervals Δx and Δt are chosen so that

$$\alpha + \beta = \frac{k\Delta t}{(\Delta x)^2} + \frac{k\Delta t}{(\Delta y)^2} \leqq \frac{1}{2}.$$

In the above discussion we assumed that the prescribed temperature is $= 0°$C on the thin line portion of the boundary. If instead, nonvanishing temperatures are

prescribed on this portion, the criterion of stability is the same as before, since the error vector satisfies (1*) and evidently is $= 0$ on the portion of the boundary in question.

We shall now discuss the modifications in the above analysis if the prescribed temperature gradient does not vanish. Let the above boundary conditions (5), (6) and (7) be replaced by

$$\frac{\partial T}{\partial y} = \phi_1(x, t) \qquad c_1 \leqq x \leqq a, y = 0 \tag{10}$$

$$\frac{\partial T}{\partial x} = \Phi(y, t) \qquad x = a, 0 \leqq y \leqq b \tag{11}$$

$$\frac{\partial T}{\partial y} = \phi_2(x, t) \qquad c_2 \leqq x \leqq a, y = b. \tag{12}$$

The difference analogs of the last three equations are

$$T_{h,0,n} = T_{h,1,n} + \Delta y \cdot \phi_1(h\Delta x, n\Delta t) \qquad c_1/\Delta x \leqq h \leqq a/\Delta x \tag{10*}$$

$$T_{M+1,k,n} = T_{M,k,n} - \Delta x \cdot \Phi(k\Delta y, n\Delta t) \qquad 1 \leqq k \leqq N \tag{11*}$$

$$T_{h,N+1,n} = T_{h,N,n} - \Delta y \cdot \phi_2(h\Delta x, n\Delta t) \qquad c_2/\Delta x \leqq h \leqq a/\Delta x \tag{12*}$$

If the difference equation (1*) is applied to the lattice points for which the last three equations hold, we ultimately get

$$\begin{aligned} T_{h,1,n+1} &= \alpha T_{h-1,1,n} + (1 - 2\alpha - 2\beta)T_{h,1,n} + \alpha T_{h+1,1,n} \\ &\qquad + \beta T_{h,2,n} + U_{h,1,n} \qquad c_1/\Delta x \leqq h \leqq a/\Delta x \end{aligned} \tag{13}$$

$$\begin{aligned} T_{M,k,n+1} &= \beta T_{M,k-1,n} + \alpha T_{M-1,k,n} + (1 - 2\alpha - 2\beta)T_{M,k,n} \\ &\qquad + \beta T_{M,k+1,n} + U_{M,k,n} \qquad 1 \leqq k \leqq N \end{aligned} \tag{14}$$

$$\begin{aligned} T_{h,N,n+1} &= \beta T_{h,N-1,n} + \alpha T_{h-1,N,n} + (1 - 2\alpha - 2\beta)T_{h,N,n} \\ &\qquad + T_{h+1,N,n} + U_{h,N,n} \end{aligned} \tag{15}$$

where we have put

$$\begin{aligned} U_{h,1,n} &= \beta \Delta y \cdot \phi_1(h\Delta x, n\Delta t) \qquad && c_1/\Delta x \leqq h \leqq a/\Delta x \\ U_{M,k,n} &= -\alpha \Delta x \cdot \Phi(k\Delta y, n\Delta t) && 1 \leqq k \leqq N \\ U_{h,N,n} &= -\beta \Delta y \cdot \phi_2(h\Delta x, n\Delta t) && c_2/\Delta x \leqq h \leqq a/\Delta x. \end{aligned} \tag{16}$$

The system of MN equations obtained by applying (1*) to the MN lattice points may be written in the form

$$\mathbf{T}_{n+1} = A\mathbf{T}_n + \mathbf{U}_n \tag{17}$$

where $\mathbf{U}_n$ is a vector whose non-vanishing components are defined in (16) and whose remaining components are $= 0$. Since the error vector $\mathbf{E}_n$ satisfies the difference equation (17) it is readily seen that

$$\mathbf{E}_{n+1} = A^n\mathbf{E}_0 + A^{n-1}\mathbf{U}_0 + A^{n-2}\mathbf{U}_1 + \cdots + \mathbf{U}_{n-1}. \tag{18}$$

From (18) it follows at once that the criterion for the stability of (17) is identical with that for (9), namely that Δx and Δy must be chosen so that $\alpha + \beta \leqq \frac{1}{2}$.

It may be briefly mentioned that the above analysis may be extended to the more general case of the boundary conditions $pT + q(\partial T/\partial n) = F(t)$ where p and q take on prescribed values along the boundary. It may also be mentioned that the above analysis may be extended to problems with cylindrical and spherical symmetry.

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High Precision Calculation of Arcsin x, Arccos x, and Arctan x

By I. E. Perlin and J. R. Garrett

1. Introduction. In this paper a polynomial approximation for Arctan x in the interval $0 \leqq x \leqq \tan \pi/24$, accurate to twenty decimal places for fixed point routines, and having an error of at most 2 in the twentieth significant figure for floating point routines is developed. By means of this polynomial Arctan x can be calculated for all real values of x. Arcsin x and Arccos x can be calculated by means of the identities:

$$\text{Arctan}\,\frac{x}{\sqrt{1-x^2}} = \text{Arcsin}\,x = \frac{\pi}{2} - \text{Arccos}\,x.$$

2. Polynomial Approximation for Arctan x. A polynomial approximation for the Arctangent is obtained from the following Fourier series expansion, given by Kogbetliantz [1], [2] and Luke [3].

$$\text{Arctan}\,(x \tan 2\theta) = 2\sum_{i=0}^{\infty} \frac{(-1)^i(\tan\theta)^{2i+1}}{2i+1}\,T_{2i+1}(x), \tag{2.1}$$

where $T_i(x)$ are the Chebyshev polynomials, i.e., $T_i(\cos y) = \cos(iy)$. The expansion (2.1) is absolutely and uniformly convergent for $|x| \leqq 1$ and $0 \leqq \theta < \pi/4$.

An approximating polynomial is obtained by truncating (2.1) after n terms. Thus,

$$P(x \tan 2\theta) = 2\sum_{i=0}^{n-1} \frac{(-1)^i(\tan\theta)^{2i+1}}{2i+1}\,T_{2i+1}(x). \tag{2.2}$$

The truncation error is

$$|\epsilon_T| \leqq \tan 2\theta\,(\tan\theta)^{2n}\,|x|. \tag{2.3}$$

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When $x \tan 2\theta$ is replaced by M and $T_{2i+1}(x)$ are expressed in terms of x, (2.2) becomes

$$P(M) = \sum_{r=0}^{n-1} \frac{(-1)^r B_r M^{2r+1}}{2r+1} \tag{2.4}$$

where

$$B_r = (1 - \tan^2 \theta)^{2r+1} \sum_{k=0}^{n-r-1} \binom{2r+k}{k} (\tan \theta)^{2k}.$$

With a choice of $n = 9$ and $\tan \theta = \tan \pi/48$, the following polynomial approximation for Arctan x is obtained.

$$P(x) = a_1 x + a_3 x^3 + \cdots + a_{17} x^{17} \tag{2.5}$$

where

$$\begin{aligned}
a_1 &= 1.0 \\
a_3 &= -0.33333\ 33333\ 33333\ 33160\ \ 7 \\
a_5 &= 0.19999\ 99999\ 99998\ 24444\ \ 8 \\
a_7 &= -0.14285\ 71428\ 56331\ 30652\ \ 9 \\
a_9 &= 0.11111\ 11109\ 07793\ 96739\ \ 3 \\
a_{11} &= -0.09090\ 90609\ 63367\ 76370\ \ 73 \\
a_{13} &= 0.07692\ 04073\ 24915\ 40813\ \ 20 \\
a_{15} &= -0.06652\ 48229\ 41310\ 82779\ \ 05 \\
a_{17} &= 0.05467\ 21009\ 39593\ 88069\ \ 41
\end{aligned}$$

The truncation error is $|\epsilon_T| < 6 \cdot 10^{-22} |x|$.

3. Procedure for Calculation Arctan x. Subdivide the interval $(0, \infty)$ into seven intervals as follows: $0 \leqq u < \tan \pi/24$, $\tan [(2j-3)\pi/24] \leqq u < \tan [(2j-1)\pi/24]$ for $j = 2, 3, 4, 5, 6$, and $\tan 11\pi/24 \leqq u < \infty$. For $|x|$ on the first interval use (2.5). For $|x|$ on the $(j+1)$st interval, $(j = 1, 2, 3, 4, 5)$, the formula employed is

$$\text{Arctan } |x| = \frac{j\pi}{12} + \text{Arctan } t_j, \tag{3.1}$$

where

$$t_j = \frac{|x| - \tan \dfrac{j\pi}{12}}{1 + |x| \tan \dfrac{j\pi}{12}}$$

is used to obtain a value t_j such that $|t_j| \leqq \tan \pi/24$. Arctan t_j is calculated by (2.5) and Arctan $|x|$ from (3.1). When $|x|$ is in the seventh interval

$$\text{Arctan}\,|x| = \frac{\pi}{2} - \text{Arctan}\,\frac{1}{|x|}, \tag{3.2}$$

and

$$\frac{1}{|x|} \leqq \tan\frac{\pi}{24}.$$

The constants $\tan j\pi/24$, $(j = 1, 2, \cdots, 11)$ and $\pi/2$ are:

$\tan \pi/24 =$	0.13165	24975	87395	85347	2
$\tan \pi/12 =$	0.26794	91924	31122	70647	3
$\tan \pi/8 =$	0.41421	35623	73095	04880	2
$\tan \pi/6 =$	0.57735	02691	89625	76450	9
$\tan 5\pi/24 =$	0.76732	69879	78960	34292	3
$\tan \pi/4 =$	1.00000	00000	00000	00000	0
$\tan 7\pi/24 =$	1.30322	53728	41205	75586	8
$\tan \pi/3 =$	1.73205	08075	68877	29352	7
$\tan 3\pi/8 =$	2.41421	35623	73095	04880	2
$\tan 5\pi/12 =$	3.73205	08075	68877	29352	7
$\tan 11\pi/24 =$	7.59575	41127	25150	44052	6
$\pi/2 =$	1.57079	63267	94896	61923	1.

4. Error Analysis.

A. General.

Errors arising from calculations by a computer may be classified into three categories according to Householder [4], namely: (1) truncation errors, (2) propagated errors, and 3) round-off errors. For the propagated error, if x and y are approximated by x' and y', respectively, and the errors in each are denoted by $\epsilon(x)$ and $\epsilon(y)$, then:

$$|\epsilon(x \pm y)| \leqq |\epsilon(x)| + |\epsilon(y)|,$$

$$\left|\frac{\epsilon(xy)}{x'y'}\right| \leqq \left|\frac{\epsilon(x)}{x'}\right| + \left|\frac{\epsilon(y)}{y'}\right|, \qquad x'y' \neq 0$$

$$\left|\frac{\epsilon\left(\frac{x}{y}\right)}{\frac{x'}{y'}}\right| \leqq \frac{\left|\frac{\epsilon(x)}{x'}\right| + \left|\frac{\epsilon(y)}{y'}\right|}{1 - \left|\frac{\epsilon(y)}{y'}\right|}, \qquad x'y' \neq 0.$$

For round-off error it is assumed that rounding is accomplished in the following manner. If λ digits are to be retained and the $(\lambda + 1)$st digit is $\geqq 5$, add one to the preceding digit; otherwise do not change the preceding digit. With this convention the round-off error in fixed point arithmetic is easily determined. For floating point arithmetic use is made of the following result. Let

$$x = (x_1\beta^{-1} + x_2\beta^{-2} + \cdots + x_\lambda\beta^{-\lambda})\beta^e, \qquad x_1 \neq 0,$$

and

$$y = (y_1\beta^{-1} + y_2\beta^{-2} + \cdots + y_\lambda\beta^{-\lambda})\beta^e, \qquad y_1 \neq 0,$$

and $x \oplus y$ represent addition, subtraction, multiplication or division. The round-off error in $x \oplus y$, $\epsilon(x \oplus y)$, is given by

$$|\epsilon(x \oplus y)| \leqq |x \oplus y| \left(\frac{\beta}{2}\right) \beta^{-\lambda}.$$

Another result useful in floating point arithmetic is the following: If $\left|\frac{x' - x}{x'}\right| < a\beta^{-\tau}$, $(\tau \geqq 1)$, then x' differs from x by at most a units in the τth significant digit. The preceding results are easily established.

B. Errors in Arctan x.

a)
$$0 \leqq x < \tan \frac{\beta}{24}.$$

For fixed point arithmetic it shall be assumed that twenty-one decimals are used. The truncation error $|\epsilon_T| < 8 \cdot 10^{-23}$. The error ϵ_p due to errors in the coefficients in (2.5) is $|\epsilon_p| < 1.2 \cdot 10^{-24}$. The round-off error ϵ_R is $|\epsilon_R| < 6.35 \cdot 10^{-22}$. Hence, the total error is less than $8 \cdot 10^{-22}$, and the calculated value of Arctan x is accurate to at least twenty decimal places.

For floating point arithmetic using twenty-one significant digits, $|\epsilon_T| < 6 \cdot 10^{-22}x$, $|\epsilon_p| < 10^{-22}x$, and $|\epsilon_R| < 1.03 \cdot 10^{-20}x$. Hence, $|\epsilon(\text{Arctan } x)| < 1.1 \cdot 10^{-20}x$, and

$$\left|\frac{\epsilon(\text{Arctan } x)}{\text{Arctan } x}\right| < 1.2 \cdot 10^{-20}.$$

Thus the calculated value of Arctan x differs from the true value by at most two units in the twentieth significant digit.

b)
$$\tan \frac{\pi}{24} \leqq x < \tan \frac{11\pi}{24}.$$

For fixed point arithmetic $|\epsilon(t_j)| < 7.8 \cdot 10^{-22}$. The propagated error in Arctan t_j due to this error in t_j is $|\epsilon(\text{Arctan } t_j)| < 7.8 \cdot 10^{-22}$. The total error in Arctan t_j is then $|\epsilon(\text{Arctan } t_j)| < 1.5 \cdot 10^{-21}$. The error in Arctan x is then $|\epsilon(\text{Arctan } x)| < 2.5 \cdot 10^{-21}$, and the calculated value of Arctan x is accurate to twenty decimal places.

For floating point arithmetic

$$\left|\frac{\epsilon(\text{Arctan } x)}{\text{Arctan } x}\right| < 7.2 \cdot 10^{-20},$$

and the calculated value of Arctan x differs from the true value by at most eight units in the twentieth significant digit.

c)
$$\tan \frac{11\pi}{24} \leqq x < \infty.$$

For fixed point arithmetic $|\epsilon(\text{Arctan } x)| < 2.3 \cdot 10^{-21}$, and hence the calculated value is accurate to twenty decimal places.

For floating point arithmetic

$$\left|\frac{\epsilon(\text{Arctan } x)}{\text{Arctan } x}\right| < 1.25 \cdot 10^{-20},$$

and hence the calculated value differs from the true value by at most two units in the twentieth significant digit.

C. Errors in Arcsin x and Arccos x.

The error for floating point arithmetic using twenty-one significant digits will be given. Arcsin x will be calculated by means of

$$\text{Arcsin } x = \text{Arctan } \frac{x}{\sqrt{1 - x^2}}.$$

The quantity $1 - x^2$ is calculated by means of $1 - x^2 = (1 - x)(1 + x)$. Then

$$\left|\frac{\epsilon(\text{Arctan } x)}{\text{Arcsin } x}\right| < 10^{-19},$$

and Arcsin x will be correct to within one unit in the nineteenth significant figure.

The error for Arccos x is similar except that a round-off error due to subtraction is introduced. This error does not affect the conclusion that Arccos x will have been obtained correctly to within one unit in the nineteenth significant figure.

5. Conclusions. From the standpoint of machine application the procedure given is economical and yields precise results. It uses only twenty stored constants; the calculation of Arctan x requires a maximum of only eleven multiplications and one division; the calculation of Arcsin x and Arccos x requires an additional multiplication, division, and square root.

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The Calculation of Toroidal Harmonics

By A. Rotenberg

1. Introduction. It is the purpose of this note to describe the mathematical techniques employed in a code [5] for the IBM 704 to calculate toroidal harmonics (associated Legendre functions of half-integral order). We use recurrence techniques similar to those used by Goldstein and Thaler [1] in calculating Bessel func-

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tions. The functions of the first kind, $P_{n-\frac{1}{2}}^m(x)$, and the second kind, $Q_{n-\frac{1}{2}}^m(x)$, both obey the recurrence relation

$$(1)\qquad (\nu - m)R_\nu^m(x) = (2\nu - 1)xR_{\nu-1}^m(x) - (\nu + m - 1)R_{\nu-2}^m(x)$$

where we have written $\nu = n - \frac{1}{2}$ and $R_\nu^m(x)$ may be either $P_\nu^m(x)$ or $Q_\nu^m(x)$. Eq. (1) holds for any type of Legendre function. The toroidal harmonics are characterized by $x \geqq 1$ and m and n integral. Since $R_{n-\frac{1}{2}}^m(x) = R_{-n-\frac{1}{2}}^m(x)$, the computations are carried out for $n \geqq 0$ only.

If two key values $P_\nu^m(x)$ and $P_{\nu+1}^m(x)$ are given, eq. (1) can be used to generate $P_{\nu+k}^m(x)$ with good accuracy since $P_{\nu+k}^m(x)$ increases with increasing k. However, the $Q_\nu^m(x)$ decrease with increasing order and it is necessary to recur in the direction of decreasing order. The method of finding the key values is described in the next section.

2. Method of Calculation. Many formulas exist for the calculation of $P_\nu^m(x)$ [2], [3]. One which is well suited for machine computation is

$$(2)\quad P_\nu^m(x) = \frac{\Gamma(\nu + m + 1)}{m!\Gamma(\nu - m + 1)}\left(\frac{x-1}{x+1}\right)^{m/2}\frac{F(-\nu, m-\nu, m+1; (x-1)/(x+1))}{[(x+1)/2]^{-\nu}}$$

where $F(a, b, c; z)$ is the hypergeometric function which can be computed from the series expansion [3]:

$$(3)\qquad F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n.$$

Since a and b are half-integral and c is an integer, the calculation is straightforward and the series converges very rapidly for all moderate values of x. Since the first term in the series is unity and the series becomes monotonically decreasing after a finite number of terms, the computation continues until the next added term is less than some small value ϵ. $P_{-\frac{1}{2}}^m$ and $P_{\frac{1}{2}}^m$ are chosen as the key values and the recurrence relation is used to get functions of higher order.

The calculation of the function of the second kind is somewhat more involved and a method suggested by Goldstein and Thaler [1] is used. $Q_{\nu+1}^m(x)$ is set to zero and $Q_\nu^m(x)$ to a small value δ. Then the recursion relation is used to get the functions of lower order. It is necessary to begin computing with functions of fairly high order to get good accuracy. It is difficult to give an exact prescription, but for small values of x one should begin with the function of order 2ν to get order ν accurately. For values of $x > 5$, the functions increase very rapidly with decreasing order and only a few terms greater than ν are needed.

The values of $Q_\nu^m(x)$ obtained by recursion are not properly normalized, and normalization can be effected by computing $Q_{-\frac{1}{2}}^m(x)$ say, from the formula [2]:

$$(4)\qquad \begin{aligned} Q_\nu^m(x) = {} & \frac{\sqrt{\pi}\Gamma(\nu + m + 1)}{2^{\nu+1}(\nu + \frac{1}{2})!}\left(\frac{x^2}{x^2-1}\right)^{m/2}\frac{1}{x^{\nu+1}} \\ & \cdot F\left(\frac{\nu - m + 1}{2}, \frac{\nu - m + 2}{2}, \frac{2\nu + 3}{2}; \frac{1}{x^2}\right) \end{aligned}$$

and normalizing the other computed functions to this value. For this normalization all values of $Q_\nu^m(x)$ are positive. Many authors include a factor $(-1)^m$ on the right

side of eq. (4). The hypergeometric function appearing in eq. (4) can be computed as described above but convergence is slow for values of x close to unity, and in this case it may be desirable to compute the functions differently [4].

A subroutine [5] has been written for the IBM 704 to calculate the toroidal harmonics using the methods described here. For given m and x a table is obtained of the first twenty values of $P^m_{n-\frac{1}{2}}(x)$ and $Q^m_{n-\frac{1}{2}}(x)$ i.e. $n = 0, 1, \cdots, 19$ as well as their derivatives using the formula [2]:

$$(5) \qquad (x^2 - 1)\frac{d}{dx}R^m_{n-\frac{1}{2}}(x) = (n - m + \tfrac{1}{2})R^m_{n+\frac{1}{2}}(x) - x(n + \tfrac{1}{2})R^m_{n-\frac{1}{2}}(x)$$

Because the functions increase very rapidly with both m and x, it is convenient to make the restriction $x < 40$, $m \leqq 21$. Where tabulated values exist [2], the code is found to give full agreement. In some other cases, the equation for the Wronskian [3] of the solutions was checked and found to be very accurately satisfied. The code computes correctly the toroidal harmonics to at least six significant figures.

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2. NBS Mathematical Tables Project, *Tables of the Associated Legendre Function*, Columbia University Press, New York, 1945.
3. P. M. Morse & H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill Book Company, New York, 1953.
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5. A. Rotenberg, *NU ASLG, Share Distribution 711*, Share Program Librarian, IBM, 590 Madison Avenue, New York 22, New York.

Transcendental Equation for the Schrödinger Equation

By J. R. M. Radok

The problem of the determination of the energy levels of single particles in cylindrical wells of different dimensions reduces to the transcendental equation

$$\mathfrak{h}_\lambda^{(1)}(i - \sqrt{k^2 - \epsilon^2}) + \mathfrak{y}_\lambda(\epsilon) = 0$$

for the Schrödinger equation, where $\mathfrak{h}_\lambda^{(1)}$, $\mathfrak{y}_\lambda$ are modified quotient Bessel functions for which a table has been published recently by Morio Onoe [1] and the variables

$$k^2 = \frac{2mUa^2}{\hbar^2}, \qquad \epsilon^2 = 2m\frac{(E + U)a^2}{\hbar^2}$$

involve the quantities

$$m = \text{mass},$$

$$U = \text{potential energy},$$

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TABLE 1

	λ = 0					λ = ±1					λ = ±2				λ = ± 3			
k	c^k_{10}	c^k_{20}	c^k_{30}	c^k_{40}	c^k_{50}	c^k_{11}	c^k_{21}	c^k_{31}	c^k_{41}	c^k_{51}	c^k_{12}	c^k_{22}	c^k_{32}	c^k_{42}	c^k_{13}	c^k_{23}	c^k_{33}	c^k_{43}
1	0.97	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
2	1.52	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
3	1.80	—	—	—	—	2.71	—	—	—	—	—	—	—	—	—	—	—	—
4	1.95	3.94	—	—	—	2.99	—	—	—	—	3.90	—	—	—	—	—	—	—
5	2.03	4.41	—	—	—	3.13	—	—	—	—	4.16	—	—	—	—	—	—	—
6	2.10	4.61	—	—	—	3.23	5.75	—	—	—	4.31	—	—	—	5.32	—	—	—
7	2.13	4.77	—	—	—	3.32	5.99	—	—	—	4.43	—	—	—	5.49	—	—	—
8	2.17	4.84	7.47	—	—	3.39	6.13	—	—	—	4.51	7.28	—	—	5.60	—	—	—
9	2.20	4.93	7.63	—	—	3.45	6.24	8.82	—	—	4.60	7.45	—	—	5.70	8.56	—	—
10	2.21	4.99	7.78	—	—	3.49	6.32	9.04	—	—	4.68	7.58	—	—	5.78	8.73	—	—
11	2.23	5.02	7.84	10.50	—	3.52	6.40	9.20	—	—	4.72	7.66	10.40	—	5.84	8.83	—	—
12	2.25	5.05	7.92	10.69	—	3.56	6.46	9.30	11.89	—	4.75	7.72	10.57	—	5.89	8.92	11.72	—
13	2.27	5.08	8.00	10.80	—	3.59	6.52	9.38	12.12	—	4.78	7.78	10.68	—	5.93	9.00	11.88	—
14	2.29	5.09	8.04	10.89	13.56	3.60	6.56	9.45	12.25	—	4.80	7.83	10.75	13.52	5.96	9.05	11.99	—
15	2.30	5.10	8.09	10.98	13.71	3.62	6.60	9.51	12.36	14.98	4.81	7.88	10.82	13.68	5.99	9.10	12.07	14.84

$$E = \text{total energy},$$
$$2\pi h = \text{Planck's constant},$$
$$a = \text{radius of the well}.$$

Using this table, the first few roots have been obtained graphically and are recorded in Table 1 to three significant digits. For most practical purposes, these values should be satisfactory. If necessary, they can be improved by use of Newton's method.

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1. M. Onoe, *Tables of Modified Quotients of Bessel Functions of the First Kind for Real and Imaginary Arguments*, Columbia University Press, New York, 1958.

A Note on Factors of $n^4 + 1$

By A. Gloden

The factorizations enumerated in this note form a sequel to my published factor table [1] of integers $n^4 + 1$. They have been obtained by means of my table of solutions of the congruence $x^4 + 1 \equiv 0 \pmod p$ for primes lying between $8 \cdot 10^5$ and 10^6 [2].

The following numbers are primes:

$n^4 + 1$ for n = 912, 914, 928, 930, 936, 952, 962, 966, 986, 992, 996.

$\frac{1}{2}(n^4 + 1)$ for n = 1071, 1087, 1101, 1119, 1123, 1125, 1135, 1163, 1173, 1183.

$\frac{1}{17}(n^4 + 1)$ for n = 1562, 1726, 1732, 1834.

$\frac{1}{41}(n^4 + 1)$ for n = 1818, 1848, 1982, 2006, 2012, 2064, 2088, 2094, 2228, 2340, 2364.

$\frac{1}{73}(n^4 + 1)$ for n = 2346.

$\frac{1}{89}(n^4 + 1)$ for n = 2262, 2302, 2544, 2682.

$\frac{1}{113}(n^4 + 1)$ for n = 2468.

$\frac{1}{137}(n^4 + 1)$ for n = 2476.

$\frac{1}{233}(n^4 + 1)$ for n = 2808.

$\frac{1}{2 \cdot 17}(n^4 + 1)$ for n = 1709, 1715, 1759, 1787, 1827, 1845, 1855, 1879, 1895, 1963, 2015, 2021, 2031, 2093, 2185, 2229, 2259, 2287, 2303, 2327, 2331.

$\frac{1}{2 \cdot 41}(n^4 + 1)$ for n = 2211, 2299, 2651, 2761, 2791, 2815.

$\frac{1}{2 \cdot 73}(n^4 + 1)$ for n = 2533, 2577, 2691, 2723, 2857.

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$\frac{1}{2\cdot 89}(n^4 + 1)$ for $n = 2747, 2771, 2885.$

$\frac{1}{2\cdot 97}(n^4 + 1)$ for $n = 2669, 2683, 2749.$

New factorizations are as follows:

$$\begin{aligned}
938^4 + 1 &= 809273\cdot 956569\\
1060^4 + 1 &= 847577\cdot 1489513\\
1348^4 + 1 &= 940169\cdot 3511993\\
1512^4 + 1 &= 926617\cdot 5640361\\
1874^4 + 1 &= 914561\cdot 13485457\\
2100^4 + 1 &= 17\cdot 873553\cdot 1309601\\
2838^4 + 1 &= 868841\cdot 74663657\\
2908^4 + 1 &= 41\cdot 940369\cdot 1854793\\
\tfrac{1}{2}(1155^4 + 1) &= 830233\cdot 1071761\\
\tfrac{1}{2}(1191^4 + 1) &= 935353\cdot 1075577\\
\tfrac{1}{2}(1509^4 + 1) &= 872369\cdot 2971849\\
\tfrac{1}{2}(2635^4 + 1) &= 857569\cdot 28107577\\
\tfrac{1}{2}(2765^4 + 1) &= 908353\cdot 32173321\\
\tfrac{1}{2}(2977^4 + 1) &= 17\cdot 809041\cdot 2855393
\end{aligned}$$

The following factorization was omitted from my original table [1]:

$$\tfrac{1}{2}(2055^4 + 1) = 17\cdot 572233\cdot 916633.$$

The least integers still incompletely factored correspond to $n = 1038$ and 1229, for even and odd values of n, respectively.

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1. A. Gloden, "Table de factorisation des nombres $n^4 + 1$ dans l'intervalle $1000 < n < 3000$," Institut Grand-Ducal de Luxembourg, *Archives*, Tome XVI, Luxembourg, 1946, p. 71–88.

2. A. Gloden, *Table des Solutions de la Congruence $x^4 + 1 \equiv 0 \pmod p$ pour $800{,}000 < p < 1{,}000{,}000$*, published by the author, rue Jean Jaurès, 11, Luxembourg, 1959.

A Note on the Solution of Quartic Equations

By Herbert E. Salzer

For any quartic equation with real coefficients,

$$X^4 + AX^3 + BX^2 + CX + D = 0, \tag{1}$$

the following condensation of the customary algebraic solution is recommended as quickest and easiest for the computer to follow (no mental effort required). It works in every exceptional case.

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Denote the four roots of (1), by X_1, X_2, X_3, and X_4. With the aid of [1], solve the "resolvent cubic equation" $ax^3 + bx^2 + cx + d = 0$ *for the real root* x_1 *only*, where

$$(2) \qquad a = 1, \qquad b = -B, \qquad c = AC - 4D, \quad \text{and} \quad d = D(4B - A^2) - C^2.$$

Find

$$(3) \qquad m = +\sqrt{\tfrac{1}{4}A^2 - B + x_1}, \qquad n = \frac{Ax_1 - 2C}{4m}.$$

If $m = 0$, take $n = \sqrt{\tfrac{1}{4}x_1^2 - D}$ and proceed according to the following *Case I* or *Case II*, depending upon whether m is real or imaginary.

Case I: If m is real, let $(\tfrac{1}{2}A^2 - x_1 - B) = \alpha$, $4n - Am = \beta$, $\sqrt{\alpha + \beta} = \gamma$, $\sqrt{\alpha - \beta} = \delta$, and finally

$$(4\text{I}) \qquad X_1 = \frac{-\tfrac{1}{2}A + m + \gamma}{2}, \qquad X_2 = \frac{-\tfrac{1}{2}A - m + \delta}{2},$$
$$X_3 = \frac{-\tfrac{1}{2}A + m - \gamma}{2}, \quad \text{and} \quad X_4 = \frac{-\tfrac{1}{2}A - m - \delta}{2}.$$

Case II: If m is imaginary, say $m = im'$, then n is also imaginary, say $n = in'$. Let

$$(\tfrac{1}{2}A^2 - x_1 - B) = \alpha, \qquad 4n' - Am' = \beta, \qquad +\sqrt{\alpha^2 + \beta^2} = \rho,$$
$$\sqrt{\frac{\alpha + \rho}{2}} = \gamma, \qquad \frac{\beta}{2\gamma} = \delta,$$

and finally

$$(4\text{II}) \quad \begin{cases} X_1 = \dfrac{-\tfrac{1}{2}A + \gamma + i(m' + \delta)}{2}, \\ \qquad\qquad X_2 = \bar{X}_1, \text{ the complex conjugate of } X_1, \\ X_3 = \dfrac{-\tfrac{1}{2}A - \gamma + i(m' - \delta)}{2} \\ \qquad\qquad \text{and } X_4 = \bar{X}_3, \text{ the complex conjugate of } X_3. \end{cases}$$

If $\gamma = 0$, we must have $\alpha = -\alpha'$, $\alpha' \geqq 0$, and formula (4II) still holds provided that in it we replace δ by $+\sqrt{\alpha'}$.

As an example consider the quartic equation $X^4 + X^3 + X^2 + X + 1 = 0$, where $A = B = C = D = 1$, so that from (2) the resolvent cubic equation is $x^3 - x^2 - 3x + 2 = 0$. From [1] we find $x_1 = 0.61803\ 400$. From (3), $m = +\sqrt{-0.13196\ 600} = +0.36327\ 125i$, so that $m' = +0.36327\ 125$. Then $n = \dfrac{-1.38196\ 600}{1.45308\ 500i} = +0.95105\ 655i$, so that $n' = +0.95105\ 655$. Proceeding according to *Case II*, $\alpha = -1.11803\ 400$, $\beta = 3.44095\ 495$, $\rho = 3.61803\ 41$, $\gamma = 1.11803\ 40$ and $\delta = 1.53884\ 18$. Then from (4II) we obtain $X_1 = 0.30901\ 70 + 0.95105\ 65i$, $X_2 = \bar{X}_1 = 0.30901\ 70 - 0.95105\ 65i$, $X_3 = -0.80901\ 70 - 0.58778\ 53i$ and

$X_4 = \bar{X}_3 = -0.80901\ 70 + 0.58778\ 53i$. These roots may be verified as correct, since they are known to be the complex fifth roots of unity, namely $X_1 = \cos 72° + i \sin 72°$, $X_2 = \cos 288° + i \sin 288°$, $X_3 = \cos 216° + i \sin 216°$, and $X_4 = \cos 144° + i \sin 144°$.

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1. H. E. Salzer, C. H. Richards & I. Arsham, *Table for the Solution of Cubic Equations*, McGraw-Hill, New York, 1958.

A Conjugate Factor Method for the Solution of a Cubic

By D. A. Magula

1. Introduction. This paper gives a simple method for computing the real roots of the reduced cubic equation with real coefficients,

$$x^3 + Ax + B = 0, \tag{1}$$

having roots a, b, c. We assume a to be real, since every cubic equation has at least one real root.

The method consists in factoring B, and setting one factor equal to $\pm\sqrt{m}$, the other n. For all pairs m, n such that $m + n = -A$, $\pm\sqrt{m}$ is a root. If no such pair exists, a method of interpolation is shown.

2. Proof of Method. The reduced cubic equation (1) can be transformed, by using the relations between the roots and coefficients, into a complete cubic,

$$p^3 + 6Ap^2 + 9A^2p + 4A^3 + 27B^2 = 0, \tag{2}$$

where

$$p = (-3a^2 - 4A). \tag{3}$$

Equation (2) can be written in the form:

$$(p + A)^2(-p - 4A) = 27B^2 \tag{4}$$

or

$$\frac{(p + A)}{3}\sqrt{\frac{(-p - 4A)}{3}} = \pm B. \tag{5}$$

Let

$$m = \frac{-p - 4A}{3} \quad \text{and} \quad n = \frac{p + A}{3} \tag{6}$$

$$n\sqrt{m} = \pm B \tag{7}$$

and

$$m + n = -A. \tag{8}$$

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From (7) and (8)

$$\sqrt{m}(-A - m) = \pm B$$

or squaring both sides:

$$m^3 + 2Am^2 + A^2m - B^2 = 0. \tag{9}$$

But the equation (9) is identical with the equation (1) whose roots are squared. Therefore,

$$m = a^2. \tag{10}$$

Thus one root of the given cubic is $\pm\sqrt{m}$. The other two roots can be found by solving the quadratic equation resulting from the depression of the given cubic:

$$x^2 + ax + (A + a^2) = 0. \tag{11}$$

3. Steps in Computation. The calculation can be carried out in the following simple operations. A model example is given: $x^3 - 7x - 6 = 0$.

A. Make a table with pairs of integral factors of B putting one in the first column, $\pm\sqrt{m}$, and the other in the second, n. In each case where $m + n = -A$, the corresponding entry in the first column is a root.

$\pm\sqrt{m}$	n	m	$m + n$
1	−6	1	−5
2	−3	4	1
3	−2	9	7
6	−1	36	35
−1	6	1	7
−2	3	4	7

In this example the equation has three integer roots which appear circled in the first column.

B. If the equation has *one* integer root, it can be found by the above method. Then the other two can be found by solving equation (11).

C. If the equation has no integer roots, we may use this method to interpolate. For example, suppose we have the equation, $x^3 - 3 = 0$, in which $A = 0$. We find from the corresponding table that the root is between 1 and 3. We can then make additional entries in the table by taking values within this interval.

$\pm\sqrt{m}$	n	m	$m + n$
1	−3	1	−2
3	−1	9	+8
1.4	−2.14	1.96	−.18
1.5	−2	2.25	+.25
1.44	−2.08	2.07	−.01
1.45	−2.07	2.10	+.03

Thus the real root of this equation is between 1.44 and 1.45, and this process can be repeated until the desired degree of accuracy is attained. From (B) it follows that the presence of imaginary roots in no way affects the solution of the cubic by this method.

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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

42[F].—A. GLODEN, *Table des Solutions de la Congruence* $x^4 + 1 \equiv 0$ *(mod p) pour* $800\,000 \leqq p \leqq 1\,000\,000$, published by the author, rue Jean Jaurès, 11, Luxembourg, 1959, 22p., 30 cm., mimeographed. Price 150 Belgian francs.

This volume represents the culmination of independent efforts of table-makers such as Cunningham, Hoppenot, Delfeld, and the author, extending over a period of several decades. The foreword contains an extensive list of references to earlier publications of this type, which combined with the tables under review give the two least positive solutions of the congruence $x^4 + 1 \equiv 0 \pmod p$ for all admissible primes less than one million. [See RMT **109,** *MTAC*, v. 11, 1957, p. 274 for a similar list of references.]

Professor Gloden has used such congruence tables in the construction of manuscript factor tables of integers $N^4 + 1$, which now extend to $N = 40\,000$, with certain omissions. The bibliography in the present set of tables also contains references to this work. Numerous references to these factor tables are also listed in RMT **2,** *MTAC*, v. 12, 1958, p. 63.

J. W. W.

43[G, X].—F. R. GANTMACHER, *Applications of the Theory of Matrices*, translated by J. L. Brenner, Interscience Pub., New York, 1959, ix + 317 p., 24 cm. Price $9.00.

This is a remarkable book containing material, not easily available elsewhere, related to the analysis of matrices as opposed to the algebra of matrices. In this I use the word analysis to mean broadly that part of mathematics largely dependent upon inequalities (and limits) as opposed to algebra, which depends largely on equalities.

In particular, the material in this book is directed largely toward studies of stability of solution of linear differential equations (in Chapters IV and V) and of matrices with nonnegative elements (in Chapter III).

Several aspects of the book will be useful to numerical analysts. These include some parts of the chapter on matrices with nonnegative elements, the implications of the chapters on stability of solutions of differential equations to the stability of numerical methods of solving differential equations, and (as the author points out) a numerically feasible method of finding the roots of polynomials.

Many topics included in this text are not easily available elsewhere. For example, product integration is expounded; this is an amusing version of Euler's method applied to the solution of linear first-order homogeneous differential equations. Most of the other exposition is unique in one way or another, and on the whole the book is a valuable contribution to literature.

This is a translation, augmented to some extent by bibliographic and other notes, of the second part of a successful Russian book. It is interesting to note that another publisher has announced the impending publication of translations of both parts.

The printing is good, and the reviewer noticed no serious errors. There are four

short appendices devoted to standard elementary theorems, a bibliography with seventy-two entries, and a useful index. The chapter headings follow:

Chapter I. Complex Symmetric, Antisymmetric and Orthogonal Matrices
Chapter II. Singular Bundles of Matrices
Chapter III. Matrices with Nonnegative Elements
Chapter IV. Applications of the Theory of Matrices to the Study of Systems of Linear Differential Equations
Chapter V. The Routh-Hurwitz Problem and Related Questions.

C. B. T.

44[K].—Joseph Berkson, "Tables for use in estimating the normal distribution by normit analysis," *Biometrika*, v. 44, 1957, p. 411–435.

In a quantal response assay a number of independent tests are made at each of a number of dose levels, and the result of each test is graded as "success" or "failure". If the probability of "success" at dose metameter value x is assumed to follow the "normal" law

$$P(x) = 1/\sqrt{2\pi} \int_{-\infty}^{(x-\mu)/\sigma} e^{-t^2/2}\, dt$$

the method of "normit analysis" is proposed by Berkson as a replacement for the familiar (iterative) method of "probit analysis" for estimating the parameters μ and σ.

Suppose that the dose metameter values used are $x_1, \cdots, x_k$, that n_i tests are made at level x_i, and that r_i of these tests result in success. Let

$$p_i = \begin{cases} 1/(2n_i) \text{ if } r_i = 0 \\ r_i/n_i \text{ if } 0 < r_i < n_i \\ 1 - 1/(2n_i) \text{ if } r_i = n_i, \end{cases}$$

$$X_i = X(p_i), \text{ where } X(p) \text{ is defined by the relation}$$

$$p = (1/\sqrt{2\pi}) \int_{-\infty}^{X(p)} e^{-u^2/2}\, du,$$

$$Z_i = (1/\sqrt{2\pi})\, e^{-X_i^2/2},$$

$$w_i = Z_i^2/p_i(1 - p_i).$$

The method consists of a weighted regression analysis, which is facilitated by tables which give for each p_i the corresponding values of w_i and w_iX_i.

In table 2 w_i and w_iX_i are given to 6D for $p = 0.001(0.001)0.500$. (For $p > \frac{1}{2}$, $w(p) = w(1 - p)$ and $wX(p) = -wX(1 - p)$.) For moderate n_i interpolation may be avoided by use of Table 1, which gives w_i and w_iX_i, also to 6D, for all combinations of r_i and n_i for which $1 < n_i \leqq 50$ and $0 \leqq r_i \leqq n/2$. (For $r > n/2$, $w(r, n) = w(n - r, n)$ and $wX(r, n) = -wX(n - r, n)$.) It is stated that the entries in both tables are correct to within ± 1 in the final digit.

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45[K].—I. D. J. Bross & E. L. Kasten, "Rapid analysis of 2 x 2 tables", Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 18–28.

Conventional statistical analysis of 2 x 2 tables such as

Sample	A	$\bar{A}$	
1	a	b	$N_1 = NP$
2	c	d	$N_2 = NQ$
	T		N

involves use of triple-entry tables for critical values of a. These tables are entered with N, N_1, and T, or some equivalent combination of three numbers. The body of the table then usually gives critical values for the observation a. The authors remark that the statistical test is relatively insensitive to variation in N and propose to reduce the complexity of the tabular entry to double entry by ignoring N and using only the parameters T and P. Charts I to IV inclusive present lower tail critical values for a at 5 %, 2.5 %, 1 % and .5 % levels of probability for $.1 < P < .9$ and $5 \leqq T \leqq 49$. Interchange of P and Q produces lower tail critical values for c (and by subtraction) upper tail critical values for a.

The authors claim that the approximation is good, provided P and Q are both at least .1 and T is not larger than $.2N$.

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46[K].—F. E. Clark, "Truncation to meet requirements on means," Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 527–536.

The problem under consideration is that of truncating a given distribution so that the resulting population will meet specified sampling requirements. This problem arises when one wishes to screen the output of some production process in order to reduce the risk (probability) of having lots rejected on the basis of a requirement that only those lots will be accepted for which the mean $\bar{X}$ of a random sample of n items shall, for example, exceed or be less than some value, say UAL (upper average level) or LAL (lower average level).

Methods are given for determining a single point of truncation A such that the mean $\bar{X}_A$ of a random sample from a normal population (μ, σ) screened or truncated at A will meet a specification requirement $\bar{X}_A \geqq$ LAL or $\bar{X}_A \leqq$ UAL with risk of rejection r.

Methods are also given for determining double points of truncation A and B such that a normal population (μ, σ) truncated at $X = A$ and at $X = B$ will meet the requirement LAL $\leqq \bar{X}_{AB} \leqq$ UAL with risk r.

As aids in carrying out the computations involved in the above methods, a table is included which lists values of the mean μ_{ab} and the standard deviation σ_{ab} of the standard normal population (0, 1) truncated at a and at b ($a \leqq b$). Entries are given to 4D for $a = -3.00(.25).50$ and for $b = 3.00(-.25)0$. A chart is included which contains curves of constant μ_{ab} and σ_{ab} for fixed degrees of truncation p,

where p is the proportion of the complete population which is eliminated by truncation. In this chart, a extends from -3.0 to 0.5, b extends from -0.5 to $+3$, $p = .05$, $.10(.10)1.0$, $\mu_{ab} = -1.0(.1)1.0$, $\sigma_{ab} = 0(.1).9$. A second chart contains a set of five curves for selected values of n and r to be used in determining a and p as a function of h, where $h = (\mu - \text{LAL})/\sigma$. Values of a extend from -1.4 to 0.4, h extends from -1.0 to 0.4, and p from .10 to .65.

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47[K].—W. H. CLATWORTHY, *Contributions on Partially Balanced Incomplete Block Designs with Two Associate Classes*, NBS Applied Mathematics Series, No. 47, U. S. Government Printing Office, Washington 25, D. C., 1956, iv + 70 p., 26 cm. Price $.45.

This publication contains six papers dealing with various aspects (enumeration, dualization, and tabulation) of partially balanced incomplete block designs with two associate classes, and with the construction of some new group divisible designs, triangular incomplete block designs, and Latin square type designs with two constraints. Approximately 75 new designs not contained in the monograph of Bose, Clatworthy, and Shrikhande [1] are given in the present paper. A number of theorems are proved in the six papers. Two of the theorems give bounds on the parameters v, p_{11}^1, and p_{12}^1 in terms of the parameters r, k, n_1, n_2, λ_1, and λ_2 of the partially balanced incomplete block design with two associate classes. The two theorems on the duals of partially balanced designs are useful in identifying certain partially balanced incomplete block designs.

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1. R. C. BOSE, W. H. CLATWORTHY & S. S. SHRIKHANDE, *Tables of Partially Balanced Designs with Two Associate Classes*, North Carolina Agricultural Experiment Station Technical Bulletin No. 107, 1954.

48[K].—W. J. DIXON, "Estimates of the mean and standard deviation of a normal population," *Ann. Math. Stat.*, v. 28, 1957, p. 806–809.

Four estimates of the mean in samples of N from a normal population are compared as to variance and efficiency. These are (a) median, (b) mid-range, (c) mean of the best two, (d) $\bar{X}_{)1,N(} = \sum_{i+2}^{N-1} [X_i/(N - 2)]$. The sample values are denoted $X_1 \leqq X_2 \leqq \cdots \leqq X_N$. The results for the median and mid-range are given primarily for comparison purposes, since results are well known. The mean of the best two is reported as the small sample equivalent of the mean of the 27th and 73rd percentiles.

The variance and efficiency are given to $3S$ for $N = 2(1)20$. The estimate (d) is compared to the best linear systematic statistics (BLSS) as developed in [1] and [2]. It is noted that the ratio Var (BLSS)/Var ($\bar{X}_{)1,N(}$) is never less than 0.990.

Two estimates of the standard deviation are given in Table II. One, the range, is well known. The quantity k which satisfies $E(kW) = \sigma$ is tabulated to $3D$ for $N = 2(1)20$. Denote the subranges $X_{N-i+1} - X_i$ by $W_{(i)}$ and $W_{(1)} = W$. The estimate $S' = k'(\sum W_{(i)})$, where the summation is over the subset of all $W_{(i)}$ which gives

minimum variance, is the other estimate. Table II compares variances and efficiences of these two estimates to $3S$ for $N = 2(1)20$. Also a column gives the ratio of the variance of (BLSS) as given in [2] to the variance of S' to $3D$ for $N = 2(1)20$.

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1. A. K. GUPTA, "Estimation of the mean and standard deviation of a normal population for a censored sample", *Biometrika*, v. 39, 1952, p. 260–273.
2. A. E. SARHAN & B. G. GREENBERG, "Estimation of location and scale parameters by order statistics from singly and doubly censored samples. Part I", *Ann. Math. Stat.* v. 27, 1956, p. 427–451.

49[K].—H. F. DODGE & H. G. ROMIG, *Sampling Inspection Tables*, Second Edition, John Wiley & Sons, Inc., New York, 1959, xi + 224 p., 29 cm. Price $8.00.

The first feature one notices about the second edition of the Dodge-Romig tables, as compared to the first edition, is its size. Measuring 8½″ x 11″, with a total of 224 pages, it stands in bold contrast to the pocket-sized 5½″ x 8½″, 106-page first edition,—a four-fold increase, in a sense indicative of the growth of statistical quality control since the first edition was published in 1944.

There are 60 pages of text covering an introduction, and four chapters which describe the principles, procedure for application, and the mathematics of the sampling plans. The titles of these chapters are, respectively: A Method of Sampling Inspection; Single Sampling and Double Sampling Inspection Tables; Using Double Sampling Inspection in a Manufacturing Plant; and Operating Characteristics of Sampling Plans. The remaining 158 pages include a table of contents, seven appendices, and an index. The last four appendices give the same four sets of tables as appear in the first edition; these are respectively entitled: Single Sampling Tables for Stated Values of Lot Tolerance Per Cent Defective (LTPD) with Consumer's Risk of 0.10; Double Sampling Tables for Stated Values of Lot Tolerance Per Cent Defective (LTPD) with Consumer's Risk of 0.10; Single Sampling Tables for Stated Values of Average Outgoing Quality Limit (AOQL); Double Sampling Tables for Stated Values of Average Outgoing Quality Limit (AOQL). The first three appendices are devoted to 120 pages of operating characteristic curves; these are, respectively, OC Curves for all Single Sampling Plans in Appendix 6; OC Curves for all Double Sampling Plans in Appendix 7; and OC Curves for Single Sampling Plans with $c = 0, 1, 2, 3$, and $n \leq 500$ (based on binomial probabilities).

As can be seen from the above list of contents, the largest part of the increase in size is due to the inclusion of three sets of operating characteristic curves,—two sets for the AOQL plans, and one set for a separate inventory of single sampling plans. This separate inventory has a wide enough range in sample size and acceptance numbers to provide a useful independent reference of OC curves for those who prefer to derive their own single sampling plans (charts to derive such plans have been retained from the first edition). Although separate sets of OC curves are not given for the LTPD plans, the authors point out how these may be obtained or estimated from the OC curves which are given.

It is heartening to see this inclusion of OC curves and the authors' comment

that it "has been urged over the years by a number of engineers", as well as the inclusion of a completely new chapter devoted to a discussion of the operating characteristic curves. This overt endorsement by this distinguished team in the field of quality control of the use of the OC curve to evaluate an acceptance sampling plan should impress upon quality control practitioners the importance of describing the assurance provided by a decision-making procedure in probability terms. Perhaps it will also emphasize the fact that while one may prefer to attach the label of LTPD plan, AOQL plan, or AQL plan to an acceptance sampling plan, depending on first considerations in deriving or classifying a plan, these sampling plans are one and the same as far as assurance in decision making is concerned, if they have the same operating characteristic curve.

The clear distinction made by the authors between OC curves giving the probability of lot acceptance based on lot quality as distinguished from process quality is most welcome, as this distinction is seldom clearly made. That there is a difference is often not realized; sometimes it is misunderstood and the importance of the difference exaggerated; at best it is ignored, since most often, but not always, the difference in OC curves is slight, as the authors point out. It is, however, unfortunate, in this reviewer's opinion, that the authors chose to attach the special labels of Type A and Type B to the corresponding OC curves, rather than simply to identify them as finite lot quality and infinite lot or process quality OC curves. The additional labels are not essential, and can do nothing more than add mystery to an already confused situation to the many who will not look beneath the labels. Unfortunately, also, an inaccuracy has crept into a statement with regard to these OC curves. On page 59 starting at the bottom of the first column the authors state, "When the sample sizes are a larger percentage of the lot size, the Type A OC curve will fall somewhat below the Type B curve shown on the chart, as can be seen in Fig. 4-1 where the Type A OC Curve for $N = \infty$ is identically the Type B OC curve." That the statement is inaccurate can indeed be seen from a careful examination of Fig. 4-1, since the finite lot (Type A) and infinite lot (Type B) OC curves intersect and cross. It would be better to remember that for the same sampling plan the OC curve for a finite lot is always more discriminating than the OC curve for an infinite lot.

Little need be said about the tables of sampling plans, which are well known as a result of the first edition. Derived on the principle of minimizing the total amount of inspection, sampling as well as screening of rejected lots, they are particularly suited to producers who are responsible for both the production and sampling inspection of their finished product. The sampling plans may also be used by a purchaser for acceptance inspection, but the choice of plan for this purpose should be based principally on the properties of the operating characteristic curve.

The format and typesetting, including tables and graphs, are considerably improved over the first edition. The division of each page of text into two columns also improves on readability. A minor typographical error which occurs in the second edition, but not in the first, appears on page 33, equation (2-1a), where C_N^M should be C_m^M.

The book can be highly recommended to those with modest or little mathematical background. The improvements in the second edition are sufficient to warrant its own place, along with other worthy texts, on the bookshelf of students and

practitioners of quality control who are interested in a comprehensive account of sampling inspection as well as in the procedures and tables for its application.

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50[K].—C. W. DUNNETT & R. A. LAMM, "Some tables of the multivariate normal probability integral with correlation coefficients $\frac{1}{3}$," Lederle Laboratories, Pearl River, New York. Deposited in UMT File.

The probability integral of the multivariate normal distribution in n dimensions, having all correlation coefficients equal to ρ (where necessarily $-\frac{1}{n-1} < \rho < 1$), is given by

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \left(\frac{1}{2\pi}\right)^{n/2} \frac{[1+(n-1)\rho]^{-1/2}}{(1-\rho)\frac{(n-1)}{2}} \exp\left[\frac{1+(n-2)\rho}{(1-\rho)[1+(n-1)\rho]}\right.$$

$$\left.\cdot\left\{\sum x_i^2 - \frac{2\rho}{1+(n-2)\rho}\sum_{i\neq j}\sum x_i x_j\right\}\right] dx_1 \cdots dx_n$$

This function, which we shall denote by $F_{n,\rho}(x_1, \cdots, k_n)$, has been tabulated for $\rho = \frac{1}{2}$ and $x_1 = \cdots = x_n$ by Teichroew [1]. In the present paper, we present a table for the case $\rho = \frac{1}{3}$ and $x_1 = \cdots = x_n$. The need for this table arose in connection with a multiple-decision problem considered by one of the authors [2].

In computing the table, use was made of the fact that, for $\rho \geqq 0$, $F_{n,\rho}(x_1, \cdots, x_n)$ belongs to a class of multivariate normal probability integrals which can be written as single integrals (see Dunnett and Sobel [3]), a fact which greatly facilitates their numerical computation. In this case, we have

$$F_n, \rho(x_1, \cdots, x_n) \equiv \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \left[F\left(\frac{x_i + \sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right] f(y)\, dy$$

where

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \quad \text{and} \quad F(y) = \int_{-\infty}^{y} f(y)\, dy.$$

The attached table was computed by replacing the right-hand side of this identity by the series based on the roots of Hermite polynomials described by Salzer *et al.* [4]. Those tabular values marked with an asterisk have been checked by comparison with the values obtained by applying Simpson's rule. The values checked were found to be systematically less than the Simpson's rule values by an amount which varied between .0000000 and .0000013, depending on n. This indicates that the error in the tabular values may be no more than 1 or 2 units in 6th decimal place, but further checks are required in order to substantiate this.

The table gives $F_{n,1/3}(x, \cdots, x)$ to six decimal places, with x varying from 0 to $7.0/\sqrt{3}$ in steps of $0.1/\sqrt{3}$ for $n = 1\ (1)\ 10$, and from $1.5/\sqrt{3}$ to $2.1/\sqrt{3}$ in steps of $0.01/\sqrt{3}$ for $n = 1\ (1)\ 10, 13, 18$.

AUTHORS' ABSTRACT

1. D. Teichroew, "Probabilities associated with order statistics in samples from two normal populations with equal variance," Chemical Corps Engineering Agency, Army Chemical Center, Maryland, 1955.

2. C. W. Dunnett, "On selecting the largest of k normal population means," (to be published in *Jn.*, Roy. Stat. Soc, Series B, 1960).

3. C. W. Dunnett & M. Sobel, "Approximations to the probability integral and certain percentage points of a multivariate analogue of Student's t-distribution," *Biometrika*, v. 42, 1955, p. 258.

4. H. E. Salzer, R. Zucker & R. Capuano, "Table of the zeros and weight factors of the first twenty Hermite polynomials," *Jn. Res.*, Nat. Bur. Standards, v. 48, 1952, p. 111.

51[K].—E. C. Fieller, H. O. Hartley & E. S. Pearson, "Tests for rank correlation coefficients. I," *Biometrika*, v. 44, 1957, p. 470–481.

This paper is concerned with sampling determination of the approximate distribution for $z_S = \tanh^{-1} r_S$ and $z_K = \tanh^{-1} r_K$, where r_S is Spearman's rank correlation coefficient and r_K is Kendall's rank correlation coefficient, for the case of sample of size n from a bivariate normal distribution. It is concluded that z_S and z_K are approximately normally distributed if n is not too small, with var $(z_S) \doteq 1.060/(n-3)$ and var $(r_K) \doteq 0.437/(n-4)$. Eight tables are presented. Table 1 contains 4D values of three versions of var (r_S) for $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$; one version is Kendall's approximate formula (adjusted), another is the observed value, and the third is a smoothed form of the observed value. Table 2 contains 3D values of var $(r_S)/[1 - (Er_S)^2]$ and 4D values of var $(r_K)/[1 - (Er_K)^2]$, also an average over ρ for each of these, for $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$. Table 3 contains 3D approximate theoretical and observed values for Ez_S, while Table 4 contains these values for Ez_K, where $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$; the second-order correction terms for the theoretical values are also stated to 3D. Table 5 contains 4D values of the observed variance of z_S and 3D values of its observed standard deviation, likewise for Table 6 with z_K, where $\rho = 0.1(0.1)0.9$ and $n = 10, 30, 50$. Table 7 contains values of χ^2 for goodness of fit tests of the normality of z_S and z_K for $n = 30, 50$. Table 8 contains 2D and 3D values of

$$(Ez_1 - Ez_2)/\sqrt{\text{var}\,(z_1) + \text{var}\,(z_2)}$$

for z_1 and z_2 representing the same correlation coefficient but with different ρ values ($\rho_2 = \rho_1 + 0.1$); this is for the product moment correlation coefficient, Spearman's coefficient and Kendall's coefficient with $\rho_1 = 0.1(0.1)0.8$ and $n = 10, 30, 50$.

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52[K].—G. Horsnell, "Economical acceptance sampling schemes," Roy. Stat. Soc., *Jn.*, sec. A., v. 120, 1957, p. 148–201.

This paper is concerned with acceptance sampling plans designed to minimize the effective cost of accepted items produced under conditions of normal production. Effective cost per accepted item is defined to be the production cost per lot plus the average cost of inspection per lot when apportioned equally over the average number of items accepted per lot from production of normal quality. Single-sample plans are examined in detail. Double-sample plans are considered briefly.

An appendix contains thirty-one separate tables for single-sampling plans,

ten of which are applicable when inspection is non-destructive and the remaining twenty-one are applicable when inspection is destructive. The table for non-destructive inspection displays c_m/c_s which is the ratio of manufacturing cost to inspection cost; n, the number of items to be sampled; k, the accepted number; $A(n, k)$, the probability of acceptance; and c/c_m, which is the ratio of effective cost to manufacturing cost. In the tables for destructive inspection, $A(n, k)$ is replaced by $A'(n, k)$, which is the expected number of accepted items per lot.

Plans for non-destructive inspection are given only for a nominal lot size of 10,000. Plans for destructive inspection are given for lot sizes of 10,000 and 20,000. For non-destructive inspection, the process average $p_0 = .01(.01).04$, the consumer's risk point $p_1 = .03(.01).07, .09$; at which the consumer's risks are .05 and .01. For destructive inspection $p_0 = .01$ and $.02$; $p_1 = .03(.01).06$, and consumer's risks are .05 and .01. The tables, however, do not include all possible combinations of the above listed parameter values.

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53[K].—N. L. JOHNSON, "Optimal sampling for quota fulfillment," *Biometrika*, v. 44, 1957, p. 518–523.

This article contains two tables to assist with the problem of obtaining a preset quota m_i of individuals from each of k strata by selecting first a sample N of the whole population and then completing quotas by sampling from separate strata. Individual cost in the first case is c and in the second c_i. Table I gives for $m_i = m$ optimal values of N for $k = 2(1)10$; $mk \doteq 50, 100, 200, 500$; $d = c_i/c = 1.25, 1.5(.5)3.0$; $d' = c_i'/c = .9, .7, .25, 0$. Here c_i' is the worth of first sample individuals in excess of quota. The tabulated values of N are solutions of the equation $Pr(N_i < m) = (c - c_i')/(c_i - c_i)$.

Table 2 gives ratio of minimized cost to cost of choosing the whole sample by sampling restricted to each stratum. This quantity is

$$\frac{1}{d} + \left(1 - \frac{d'}{d}\right)\left(1 - \frac{1}{k}\right)^{N+1} \binom{N}{m} (k - 1)^{-m}$$

and is tabulated for $k = 2(1)5, 10$; $km \doteq 50, 100, 500$; $d = 1.5, 2.5, 3$; $d' = .5, .1, 0$.

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54[K].—P. G. MOORE, "The two-sample t-test based on range," *Biometrika*, v. 44, 1957, p. 482–489.

This paper provides a sample statistic for unequal sample sizes for a two-sample t-test based on observed sample ranges instead of sums of squares. The statistic used by the author is simply

$$u = \frac{|\bar{x}_1 - \bar{x}_2|}{w_1 + w_2},$$

where $\bar{x}_1$ and $\bar{x}_2$ are the sample means, and w_1 and w_2 are the sample ranges. Since unequal sample sizes are now permitted, the mean range $(w_1 + w_2)/2$ proposed by Lord [1] in his original paper is no longer used. Moore shows that there is very little loss in power resulting from the use of the simple sum $w_1 + w_2$, rather than a weighted sum of sample ranges, although for the estimation of σ separately, he gives a table of $f(n_1, n_2)$ and $d_{n_1} + fd_{n_2}$ to $3D$ for $n_1, n_2 = 2(1)20$ to estimate σ from

$$g = \frac{w_1 + fw_2}{d_{n_1} + fd_{n_2}},$$

which minimizes the coefficient of variation of the range estimates of population standard deviation.

The main use of this paper is, of course, the tables of percentage points (10%, 5%, 2%, and 1% points) to $3D$ for the statistic u, above. The tables of percentage points were computed by making use of Patnaik's chi-approximation for the distribution of the range, which resulted in sufficient accuracy. The limits for sample sizes are $n_1, n_2 = 2(1)20$, which fulfills the most usual needs in practice. With this work of Moore, therefore, the practicing statistician has available a very quick and suitably efficient procedure for testing the hypothesis of equal means for two normal populations of equal variance.

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1. E. Lord, "The use of range in place of standard deviation in the t-test", *Biometrika*, v. 34, 1947, p. 41–67.

55[K].—National Bureau of Standards, *Tables of the Bivariate Normal Distribution Function and Related Functions*, Applied Mathematics Series, No. 50, 1959, xliii + 258 p., 27cm. U. S. Government Printing Office, Washington, D. C. Price $3.25.

These tables, compiled and edited by the National Bureau of Standards, provide values for the probability content $L(h, k, r)$ of an infinite rectangle with vertex at the cut-off point (h, k) under a standardized and centered bivariate distribution with correlation coefficient r:

$$L(h, k, r) = \frac{1}{2\pi\sqrt{1 - r^2}} \int_h^\infty \int_k^\infty \exp\left[-\frac{1}{2}\left(x^2 + y^2 - 2rxy\right)/(1 - r^2)\right] dx\, dy$$

The range of tabulation is $h, k = 0(.1)4$, $r = 0(.05)0.95(.01)1$, the values of $L(h, k, r)$ being given to 6 decimal places. For negative correlations, the range of tabulation is $h, k = 0(.1)h_n, k_n$, $r = 0(.05)0.95(.01)1$, the values of $L(h, k, r)$ being given to 7 decimal places, where $L(h_n, k_n, -r) \leq \frac{1}{2}.10^{-7}$ if h_n and k_n are both less than 4. The two tables of $L(h, k, r)$ for positive and negative r, respectively (Tables I and II in the text), may therefore be regarded as extensions of Karl Pearson's tables of the bivariate normal distribution in his celebrated *Tables for Statisticians and Biometricians*, Part II, since the range of parameters in the latter tables is $h, k = 0(.1)2.6$, $r = -1(.05)1$. In this connection, the authors of the

present tables present a list of 31 errors in Pearson's tables, together with the corresponding correct values.

Table III gives the values to 7 decimal places, with the last place uncertain by 2 units, of the probability content, $V(h, \lambda h)$, of a certain right-angled triangle under a centered circular normal distribution with unit variance in any direction. The triangle has one vertex at the center of the distribution, with the angle at that vertex arc tan λ, while the lengths of the two bounding sides at the vertex are h and $h\sqrt{1 + \lambda^2}$. Formally,

$$V(h, \lambda h) = \frac{1}{2\pi} \int_0^h dx \int_0^{\lambda x} \exp\left[-\frac{1}{2}(x^2 + y^2)\right] dy.$$

The range of tabulated values for $V(h, \lambda h)$ is $h = 0(.01)4(.02)4.6(.1)5.6$ and ∞, and $\lambda = 0.1(.1)1$. Table IV gives the values of

$$V(\lambda h, h) = \frac{1}{2\pi} \int_0^{\lambda h} dx \int_0^{x/\lambda} \exp\left[-\frac{1}{2}(x^2 + y^2)\right] dy$$

for the parameters $h = 0(.01)4(.02)5.6$ and ∞, and $\lambda = 0.1(.1)1$, the degree of accuracy being the same as for the values of $V(h, \lambda h)$ in Table III.

Finally, a short table (Table V) for values of $y = \text{arc sin } r/2\pi$, $r = 0(.01)1$, correct to 8 decimal places is provided.

The two-parameter function V is related to the three-parameter function L by the formula

$$L(h, k, r) = V\left(h, \frac{k - rh}{\sqrt{1 - r^2}}\right) + V\left(k, \frac{h - rk}{\sqrt{1 - r^2}}\right) + F,$$

where

$$F = \tfrac{1}{4}[1 - \alpha(h) - \alpha(k)] + y$$

and

$$\alpha(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \exp\left(-\frac{1}{2}t^2\right) dt.$$

This relationship enables $L(h, k, r)$ to be computed to 6-decimal accuracy from the 7-decimal values of $V(h, \lambda h)$ in regions where interpolation in $L(h, k, r)$ is difficult.

The function V is also of considerable intrinsic interest, and finds applications in such fields as the probability content of polygons when the underlying distribution is bivariate normal, the distribution of range for normal samples of size 3, the non-central t-distribution for odd degrees of freedom, and one-dimensional heat flow problems. These applications are illustrated clearly and in detail in the section of the book headed Application of the Tables, due to Dr. D. B. Owen. The latter section also provides illustrations of the use of $L(h, k, r)$ in problems ranging from measurement errors, calibration systems, double-sample test procedures, and percentage changes in sample means to some interesting problems in selection (involving the correlation between aptitude test and job performance) and estimation of correlation. Finally, an introductory section discusses the mathematical properties of the L- and V-functions as well as methods of interpolation in the tables.

The authors are to be heartily commended for this most useful book, which will

place many statisticians, both practising and those more theoretically inclined, in their debt. The visual appearance and general presentation of the material are excellent. Perhaps one very minor flaw is that since $L(h, k, r) = L(k, h, r)$, tables of $L(h, k, r)$ for $h \geqq k$ would have been sufficient. However, this is a flaw (if at all) from the point of view of economics, but hardly so from the point of view of the user of the tables! The cost of the book is remarkably low.

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56[K].—E. NIEVERGELT, "Die Rangkorrelation U," *Mitteilungsblatt für Math. Stat.*, v. 9, 1957, p. 196–232.

In contrast to Spearman's rank correlation R and Kendall's coefficient T, the author studies the van de Waerden coefficient U. Let p_i and q_i $(i = 1, 2, 3 \cdots n)$ be the ranks of n observations on two variables x and y, and let ξ_i, η_i, ζ_i be the inverses of the normal probabilities: $F(\xi_i) = p_i/(n + 1)$; $F(\eta_i) = q_i/(n + 1)$; $F(\zeta_i) = i/(n + 1)$. Then, U is defined by $U = \sum_{i=1}^{n} \xi_i\eta_i / \sum_{i=1}^{n} \zeta_i^2$.

If x and y are independent the expectation of U is zero and its standard deviation is $\sigma_U = (n - 1)^{-1/2}$, as for Spearman's coefficient. The author calculates also the 4th and 6th moments of U and R, which differ. For n large, U is asymptotically normally distributed about mean zero with standard deviation σ_U. The distribution of U (to $4D$) is tabulated completely for $n = 4$, and over the upper 5% tail for $n = 5, 6, 7$. For larger values of n the Gram-Charlier development is used. Tables testing independence based on 5%, 2.5%, 1%, and .5% probabilities are given to $3D$ for $n = 6(1)\,30$. For $n > 30$ the normal probability function can be used.

In the case of dependence the correlation between R and U decreases slowly with n increasing. If x and y are normally distributed with zero mean, unit standard deviation and correlation ρ a generalization U^* of U to the continuous case leads to $U^* = \rho$. A consistent estimate for ρ is given. The U test is more powerful than the R test.

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57[K].—D. B. OWEN & D. T. MONK, *Tables of the Normal Probability Integral*, Sandia Corporation Technical Memorandum 64-57-51, 1957, 58 p., 22 x 28 cm. Available from the Office of Technical Services, Dept. of Commerce, Washington 25, D. C., (Physics (TID-4500, 13th Edn.), Price \$.40.

The following forms of the normal probability integral

$$G(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h} e^{-t^2/2}\, dt, \; h \geqq 0$$

$$G(-h) = 1 - G(h)$$

are given for $h = 0(.001)\,4(.01)\,7$, to $8D$. For those having frequent use of $G(h)$ these tables eliminate the simple yet troublesome computation necessary when

using the more comprehensive tables prepared at the New York Mathematical Tables Project [1] for

$$F(h) = \frac{1}{\sqrt{2\pi}} \int_{-h}^{h} {}^{-t^2/2}\, dt$$

The present tables are more comprehensive than the Pearson and Hartley tables [2] for $G(h)$ up to $h = 7$. These tables have been checked by the authors against other tables. (The reviewer could undertake no systematic checking, but such occasional checks as were made revealed no errors.)

Computations (made on a CRC-102A digital computer) were facilitated by using the following continued fractions:

$$R_1(x) = \cfrac{x}{1 - \cfrac{x^2}{3 + \cfrac{2\,x^2}{5 - \cfrac{3\,x^2}{7 + \cdots}}}}, \quad h \leqq 2.5;$$

$$R_2(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cdots}}}}, \quad h \geqq 2.5.$$

Then $G(h)$ was computed from

$$G(h) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-h^2/2} R_1(h), \quad \text{or} \quad 1 - \frac{1}{\sqrt{2\pi}} e^{-h^2/2} R_2(h).$$

For large values of h, $G(-h)$ can be obtained from the formula

$$G(-h) = \frac{1}{\sqrt{2\pi}} e^{-h^2/2} M(h), \qquad h \geqq 0,$$

where

$$M(h) = e^{-h^2/2} \int_h^\infty e^{-t^2/2}\, dt.$$

$M(h)$, Mill's ratio [3], provides constants a, an integer, and b $(0.1 < b < 1)$ such that

$$G(-h) = b.10^{-a}$$

For large $h = 50(1)\,150(5)\,500$, $M(h)$ and b are tabulated to $8S$; a, of course, exactly.

The authors checked $-\log_{10} G(-h)$ given in [2] for large h, and found one discrepancy, namely, $-\log_{10} G(-500)$ should read 54289.90830.

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1. *Tables of Normal Probability Functions*, National Bureau of Standards Applied Math. Series, No. 23, U. S. Government Printing Office, Washington, D. C., 1953.

2. E. S. PEARSON, H. O. HARTLEY, *Biometrika Tables for Statisticians*, v. 1, Cambridge University Press, London, 1954, p. 104–111.
3. R. D. GORDON, "Values of Mill's ratio of area to bounding ordinate of the normal probability integral for large values of the argument," *Ann. Math. Stat.*, v. 12, 1941, p. 364–366.

58[K].—A. E. SARHAN & B. G. GREENBERG, "Tables for best linear estimates by order statistics of parameters of single exponential distributions from singly and doubly censored samples," Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 58–87.

Tables are provided for the exact coefficients of the best linear systematic statistics for estimating the scale parameter of a one-parameter single exponential distribution and the scale and location parameters of a two-parameter single exponential distribution. All possible combinations of samples of size n with the r_1 lowest and r_2 highest values censored are considered for $n \leqq 10$. Exact coefficients for the best linear systematic statistic for estimating the mean (equal to the location parameter plus the scale parameter) are also given for the two parameter case. Other tables give the variances, exact or to $7D$, of the estimates obtained and the efficiency relative to the best linear estimate to $4D$ based on the complete sample. These extensive tables are of immediate practical importance in many fields, such as life testing and biological experimentation.

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59[K].—Y. S. SATHE & A. R. KAMAT, "Approximations to the distributions of some measures of dispersion based on successive differences," *Biometrika*, v. 44, 1957, p. 349–359.

Let $x_1, \cdots, x_n$ be a random sample from a normal population with variance σ^2 and let

$$\delta^2 = \frac{1}{n-1}\sum_{i=1}^{n-1}(x_i - x_{i+1})^2, \qquad d = \frac{1}{n-1}\sum_{i=1}^{n-1}|x_i - x_{i+1}|,$$

$$\delta_2^2 = \frac{1}{n-2}\sum_{i=1}^{n-2}(x_i - 2x_{i+1} + x_{i+2})^2, \qquad d_2 = \frac{1}{n-2}\sum_{i=1}^{n-2}|x_i - 2x_{i+1} + x_{i+2}|.$$

The problem is to develop approximations to the distributions of these four types of statistics. Let u be any one of these statistics. The method followed is to assume that u is approximately distributed as $(\chi_\nu^2/c)^\alpha$, where χ_ν^2 has a chi-square distribution with ν degrees of freedom; that is, taking $\lambda = 1/\alpha$, that cu^λ is approximately distributed as χ^2 with ν degrees of freedom. The constants c, α (or λ), and ν are then determined by equating the first three moments of u to those of $(\chi_\nu^2/c)^\alpha$. The results show that a fixed value can be used for α (or λ) if $n \geqq 5$. This allows two independent measures of variability u_1 and u_2, based on the same type of statistic, to be compared by use of the F test when $n \geqq 5$ for both statistics. The basic results of the paper are given in Table 1. There, for each of δ^2/σ^2, d/σ, δ_2^2/σ^2, and d_2/σ, fixed values are stated for λ, while $3D$ values for ν and $4D$ values for $\log_{10} c$ are given for $n = 5(1)\ 20, 25, 30, 40, 50$. Table 2 deals with an example. Table 3 lists the results of some approximations to δ^2/σ^2 by $(\chi_\nu^2/c)^\alpha$ for $n = 5, 10, 20, 30, 50$. Table 4 lists for comparison purposes, the upper and lower 1% and 5% points for four

approximations to δ^2/σ^2 when $n = 15, 20$. Table 5 is important; it contains 2D values of upper and lower 0.5%, 1.0%, 2.5%, and 5% points for the approximate distribution developed for δ^2/σ^2. Table 6 lists the results of some approximations to d/σ by $(\chi_\nu^2/c)^\alpha$ for $n = 5, 10, 20, 30, 50$. Finally, Table 7 furnishes 4D values of the β_1, β_2 differences that result from using a fixed λ for the $(\chi_\nu^2/c)^\alpha$ approximation to the distribution of δ_2^2/σ^2, and from using a fixed λ for the $(\chi_\nu^2/c)^\alpha$ approximation to the distribution of d_2/σ, for $n = 5, 7, 10(5)30, 40, 50$.

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60[K].—C. C. Sekar, S. P. Agarwala & P. N. Chakraborty, "On the power function of a test of significance for the difference between two proportions," *Sankhya*, v. 15, 1955, p. 381–390.

The authors determine the power function of the following statistical test: a sample of size n is drawn from each of two binomial distributions with unspecified probabilities of success p_1 and p_2, respectively. The null hypothesis is $H_0 : p_1 = p_2 = p$. For the two-sided test (alternative hypothesis: $p_1 < p_2$ or $p_1 > p_2$) at significance level α, the critical region is determined by the following conditions:

1) For a given total number r of successes in the two samples, the conditional probability of rejection under H_0 is $\leqq \alpha$.

2) If the partition $(a, r - a)$ of r successes is contained in the critical region and $0 < a < r - a$, then the partition $(a - 1, r - a + 1)$ is contained in the critical region.

3) If the partition $(a, r - a)$ is contained in the critical region, the partition $(r - a, a)$ is contained in the critical region.

A similar definition is used for the one-sided test of H_0 against the alternative $p_1 > p_2$. The critical region is determined using the exact conditional probabilities for these partitions given by S. Swaroop, [1].

The power function for the two-sided test is given to 5D for p_1 and $p_2 = .1(.1).9$; $n = 5(5)20(10)50, 100, 200$, and for $a = .05$. For the one-sided test the power function to 5D is given for the same levels of p_1, p_2 and n, and for $\alpha = .025$.

The critical region used by the authors is the one defined for the exact test by E. S. Pearson, [2]. However, for small sample sizes the power differs considerably from Patnaik's determinations, which are based on an approximately derived critical region and which use a normal distribution approximation of the probabilities.

Examples of the use of the tables are included.

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1. Satya Swaroop, "Tables of the exact values of probabilities for testing the significance of differences between proportions based on pairs of small samples," *Sankhya*, v. 4, 1938, p. 73–84.

2. E. S. Pearson, "The choice of statistical tests illustrated on the interpretation of data classed in a 2 x 2 table," *Biometrika*, v. 34, 1947, p. 139–167.

3. P. B. Patnaik, "The power function of the test between two proportions in a 2 x 2 table," *Biometrika*, v. 35, 1948, p. 157–175.

61[K].—B. Sherman, "Percentiles of the ω_n statistic," *Ann. Math. Stat.*, v. 28, 1957, p. 259–261.

The statistic

$$\omega_n = \frac{1}{2}\sum_{k=1}^{n+1}\left| L_k - \frac{1}{n+1}\right|, \qquad 0 \leqq \omega_n \leqq \frac{n}{n+1},$$

is one of several which have been suggested in connection with the null hypothesis that x_i, $(i = 1, \cdots, n)$ is a random sample from the uniform distribution and the L_k are the lengths of the $n + 1$ subintervals of the unit interval defined by the ordered sample. In most cases of interest, the x_i are the probability transforms of observations on a random variable with a continuous distribution function. Based on the distribution function derived by the author [1], the 99th, 95th, and 90th percentiles of ω_n to $5D$ for $n = 1(1)20$ have been computed, and are given in Table I. Values of two standardized forms of this statistic (based on the exact and asymptotic mean and variance, respectively) which are asymptotically normal are given in Table II to $5S$ for the same percentiles as in Table I and for $n = 5(5)15(1)20$. The author points out that the rate of convergence to the limiting values is slow.

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1. B. Sherman, "A random variable related to the spacing of sample values," *Ann. Math. Stat.*, v. 21, 1950 p. 339–361.

62[K, X, Z].—E. D. Cashwell & C. J. Everett, *A Practical Manual on the Monte Carlo Method for Random Walk Problems*, Pergamon, 1959, 152 p., 23 cm. Price $6.00.

This is volume I of the publisher's series "International Tracts in Computer Science and Technology and Their Application". It is devoted to a direct and elementary attack on the Monte Carlo principle (that is, the principle of using simulation for calculation and recording the sample statistics obtained from the simulation) in random walk problems, such as mean free path and scatter problems. Many examples are given in the text, and an appendix is added listing twenty more or less typical problems in which the Monte Carlo method was used at the Los Alamos Scientific Laboratory.

Computational details are given and in many cases flow charts are included. No full machine codes are given, but most of the calculations were done on the MANIAC I computer at Los Alamos, and coding from the descriptions given and the flow charts is probably easier than any attempt to translate a MANIAC I code to a code suitable for another machine. A disappointingly short chapter on statistical considerations is included; the reader should be warned that this is not a suitable exposition of the theory or even the practice of the statistical handling of the statistics gathered in his simulation. However, it also is treated from a definitely computational point of view, including flow charts, and is interesting from this point of view.

An interesting small chapter titled "Remarks on Computation" is also included.

There is a section on scaling, a section on debugging, a section on special routines, a section on a Monte Carlo device for determining the square root of r, and a section on a Monte Carlo device for the cosine of an equi-distributed angle. The random number routine is the familiar routine of selecting the middle digits of the square of a quasi-random number; it is frowned on by many random-number specialists. The logarithm routine presented depends on the power series expansion of the log, with restriction of the size of the arguments to assure fast convergence. An exponential routine is given as a quadratic approximation with scaling of the argument. A cosine routine is given through the use of a trigonometric identity and a truncated power series for the sine of a related angle. There is no detailed discussion of the accuracy of any of these routines.

The exposition in this book is far from perfect, and the editors have included the following statement: "It is realized that many workers in this fast moving field cannot devote the necessary time to producing a finished monograph. Because of their concern for speedy publication, the Editors will not expect the contributions to be of a polished literary standard if the originality of the ideas they contain warrant immediate and wide dissemination." The reviewer feels that the present book in its present form is more than justified on the basis of this philosophy, and he recommends the book as a most valuable contribution to numerical analysis.

The chapter headings follow:

Chapter I. Basic Principles
Chapter II. The Source Routine
Chapter III. The Main Free Path and Transmission
Chapter IV. The Collision or Escape Routine
Chapter V. The Collision Routine for Neutrons
Chapter VI. Photon Collisions
Chapter VII. Direction Parameters After Collision
Chapter VIII. Terminal Classification
Chapter IX. Remarks on Computation
Chapter X. Statistical Considerations
Appendix. Summary of Certain Problems Run on MANIAC I.

C. B. T.

63[L].—Centre National D'Études des Télécommunications, *Tables numériques des fonctions associées de Legendre. Fonctions associées de première espece, P_n^m (cos θ)*, deuxième fascicule, Éditions de la Revue Optique, Paris, 1959, xii + 640 p., 31 cm. Price 5600 F.

The first volume of these Tables was reviewed in *MTAC*, v. 7, p. 178. The present second volume was designed to extend the range of tabulation from $\theta = 90°$ to $\theta = 180°$. In the process of constructing these tables, however, it was found desirable to increase the number of decimals and to add second and fourth central differences, thus facilitating interpolation. For this reason, the range up to $\theta = 90°$, already covered in the first volume, is included (in an improved form) in the volume under review. Perhaps because of the increase in size consequent upon increased numbers of decimals and added differences, tabulation has been restricted to $m =$

0, 1, 2 (whereas vol. 1 has $m = 0(1)5$). Otherwise, the range and intervals of this volume match those of the first.

Since P_n^m (cos θ) has a singularity at $\theta = 180°$ (except when n is an integer), an auxiliary function T_n^m (cos θ) is introduced by the relation

$$P_n^m(\cos\theta) = (\csc^m\theta)T_n^m(\cos\theta) + (-1)^m(A_n \log_{10}\left(\cot\frac{\theta}{2}\right) P_n^m(\cos(180° - \theta))$$

where $A_n = (2 \sin n\pi)/(\pi \log_{10} e)$. The function T_n^m (cos θ) is tabulated for $135° \leq \theta \leq 180°$. The use of these auxiliary functions is facilitated by the provision of tables of A_n, csc θ, $\csc^2 \theta$, and $\log_{10}$ cot $(\theta/2)$.

The introductory material contains formulas, an account of the tables, hints for interpolation, and level curves of P_n (cos θ), P_n^1 (cos θ), and P_n^2 (cos θ).

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64[L].—Louis Robin, *Fonctions sphérique de Legendre et fonctions sphéroidales*, tome 3, Gauthier-Villars, Paris, 1959, viii + 289 p., 24 cm. Price 5500 F.

The first two volumes of this work were reviewed in *MTAC*, v. 13, p. 325f. The present, final, volume contains chapters VII to X.

Chapter VII is devoted to the addition theorems of Legendre functions. Both Legendre functions of the first and second kind are included, and two cases are distinguished according as the composite argument lies in the complex plane cut from $-\infty$ to $+1$ or else on the cut between -1 and 1. Addition theorems are also developed for the associated Legendre functions of the first kind.

Chapter VIII is devoted to zeros of Legendre functions. First the zeros of $P_n^m(\mu)$ as functions of μ, for fixed real m and n are discussed, then the zeros of $P^m_{-\frac{1}{2}+i_p}(\mu)$ when m is an integer and p a fixed real number, and then the zeros of $Q_n^m(\mu)$. This chapter contains also a discussion of zeros of Legendre functions considered as functions of n, m and μ being fixed. (These zeros are of importance in certain boundary-value problems.)

In Chapter IX, applications of Legendre functions are given to partial differential equation problems relating to surfaces of revolution other than spheres. Prolate and oblate spheroidal harmonics, toroidal harmonics, and conal harmonics are discussed.

Chapter X contains the discussion of some functions related to Bessel functions, namely, Gegenbauer polynomials and functions, and spheroidal wave functions.

Appendix I summarizes relevant information on "spherical Bessel functions", and Appendix II lists numerical tables of Legendre functions and tables connected with these functions.

The third volume maintains the high standards set by the first two volumes, and the author must be congratulated upon the completion of this valuable work.

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65[L, M.].—R. G. Selfridge & J. E. Maxfield, *A Table of the Incomplete Elliptic Integral of the Third Kind*, Dover Publications, Inc., 1959. xiv + 805 p., 22 cm. Price \$7.50.

The advent of fast automatic computers has made a considerable difference to the art of making and publishing tables. It has speeded up the computing processes fantastically, without, except in relatively minor ways, modifying the labor and care needed during publication processes. It has also increased the care needed in *planning* calculations; the plan has to be quite precise and exact in all details for the machine to produce proper results, whereas in desk computation, the plan can be built up and modified as the work proceeds.

The glamour of fast computation has led quite a number of people to enter the table-making field; people who appear to imagine that the whole problem is simplified by automatic computers, who perhaps do not even realize the need to seek expert advice. It is with some reluctance, but with the feeling that it is an urgent duty that needs to be performed on general grounds, that I suggest that the table now reviewed presents one of the most deplorable examples of inadequate planning and poor execution that I have met.

The tables give entries that purport to be 6-decimal values of

$$\Pi(\phi, \alpha^2, k) = \int_0^\phi \frac{d\theta}{(1 - \alpha^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}}$$

for $k = \sin \theta, \quad \theta = .1(.1)1.5, \quad \phi = 0(.01)1.57,$

$$\alpha^2 = -1(.05) - .1(.02) - .02, .05(.05).5(.02).8(.01).99.$$

Also given are two lines for $\phi = 1.5707963$, representing a direct and a check calculation.

One example of bad planning is illustrated by the arguments θ, and the heading, erroneously labelled α. These end arbitrarily in 4 or 5 zeros or 4 or 5 nines. The latter is obviously wrong and quite intolerable in print. The present authors are not unique in exhibiting this fault, which is quite inexcusable. It is a matter of, at most, a few minutes to modify a program, on any automatic machine, to round-off at the appropriate figure and to suppress printing thereafter, giving better-looking and more convincing argument values. The authors have, in any case given special treatment to the argument θ which has one *more* decimal than function values; why not treat it properly?

Both bad planning and poor execution are exhibited by the check up. The foreword states that "the greatest difficulty was encountered not in constructing the table, but in obtaining satisfactory checking". This has not, in fact, been achieved! The method described could have been—but is not—satisfactory for $\alpha^2 < 0$ and for $0 < k^2 < \alpha^2$, but the method as described, of integrating through a singularity when $\alpha^2 > 0$, is absurdly inadequate. It is not surprising then, that the final two lines, both for $\phi = 1.5707963$ as mentioned before, should often be in disagreement. It is surprising, however, that the authors accept this as a legitimate problem to hand on to their readers. The discrepancies indicate errors; the authors' duty is to find and remove these. This states the obvious, but it is equally obviously necessary to do so. Let it be said again, that to make a program to deal with this properly

is a job to be done once and for all. It may take a little effort, time, and money, but this is not to be compared with effort, time, and money wasted if it is *not* done before publishing a book of inadequate tables.

I believe, for my part, the electronic computation is so fast and easy that a discrepancy of more than a single unit (known or unknown) is not tolerable in a published table (of which the computing effort and cost are now often only a small part of the whole effort and cost); however, I am willing to concede that discrepancies up to perhaps two units *might*, on rare occasions, be justified, provided this is clearly stated. It is quite intolerable to have the following discrepancies; to pick out some of the worst:

Page	α^2	k^2	$\Pi(\alpha^2, k)$
661	0.87	0.86869...	8.654098 and 8.654259
733	0.93	0.92844...	15.309882 and 15.310251
745	0.94	0.92844...	16.857725 and 16.857859
793	0.98	0.97111...	45.498015 and 45.498457
805	0.99	0.92844...	49.243943 and 49.244046
805	0.99	0.97111...	70.018520 and 70.018897

The discrepancies are, in fact, highly systematic throughout; they are all of the same sign, except for the violent cases mostly listed above near the singularity $k = \alpha$ mentioned in the introduction. They indicate clearly that at least one set of the check values is erroneous because of an inadequate method, and not merely because of rounding; severe doubt is cast on *both* discrepant values. Dr. J. W. Wrench, Jr. has computed anew the value on p. 733 for $\alpha^2 = .93$, $k^2 = \sin^2 1.3$, and finds 15.3098662, which is not even between the two values quoted from the book; neither published value is correct and one errs by more than the discrepancy between them. I repeat again, it is the duty of the table compiler to remove *all* these doubts.

The poor execution is also exhibited by the fact that in the heading, α and k appear in place of the correct α^2 and k^2, while the 10-decimal values of k^2 given (which are simply $\sin^2 \lambda$ for $\lambda = .1(.1)1.5$) have end figure errors running up to 11 units. Again, the argument is given as θ in the tables; this corresponds to ϕ in the introduction. It is only fair to add that the heading errors in α^2, k^2, ϕ have been announced as errata. Another awkward point for the user is that absence of values for $\alpha = 0$, and for $k = 0$, makes the tables harder to interpolate.

From all this, it is evident that the authors are lacking in experience of table-making so that their remark "With the argument as outlined, no attempt has been made to proof or check the printed sheets in any way other than by comparison of the resultant complete integrals" causes less surprise than might otherwise be the case. If there were no other faults occurring other than those mentioned above, they would have been exceptionally and unduly lucky. Photographic processes seem as far from infallibility as printing from letter-press; the possible faults are different, but nevertheless exist just the same. In fact, a rather superficial examination of the table reveals unsightly irregularities in spacing of lines on pages 51,

623, 700 and some lesser ones elsewhere. It would have been easy to reprint the pages before reproduction. More serious are several digits that are not fully legible:

p. 446	$\theta = .25$	$k = .03946 \cdots$	3rd digit 3
p. 507	$\theta = .75$	$k = .15164 \cdots$	1st digit 8
p. 579	Bottom right corner.	Very faint.	

All these imperfections occur in at least two copies of the tables; such imperfections are common in tables printed from typescript and should be expected and sought out. The real surprise is, however, as mentioned above, that, after the numerical comparison of check values mentioned had been made, its lack of success seems simply to have been ignored.

It is hoped that possible users may, with the exercise of necessary—but undue—caution, obtain adequate results, maybe 5½ correct figures, if they need them. The publication of this book will undoubtedly make it much more difficult to publish a good and proper version; this is a major criticism of such a book. The only consolation I can offer the authors is that I have seen several tables that are even worse.

As I have said, I have expressed myself so freely with some reluctance, from a sense of duty; it is no part of my desire to discourage the enthusiasm of table-makers, but they must realize the magnitude and duties of the task so taken on, and seek competent advice before proceeding with the work, and potential users must be adequately warned.

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66[S].—D. R. Hartree, *The Calculation of Atomic Structures*, John Wiley and Sons, Inc., New York, 1957, xiii + 181 p., 23 cm. Price $5.00.

This book by the late D. R. Hartree is the fruit of a lifetime of experience in the calculation of the outer, electronic structure of atoms. It is concerned with methods for the calculation of atomic structures rather than with the results of such calculations for particular atoms. Emphasis is deliberately placed on means of obtaining "best" approximations which can be both represented and applied simply. The student who wants an introduction to the essential methods of approximation and computation of shell structures may read the first hundred pages. The mathematician will find in this book the physical background for the author's well-known text on numerical analysis.

In the Introduction are outlined the seven main steps in the development of atomic theory up to the point at which quantitative calculations are possible. The atomic units are introduced and the point change approximation of the electron justified. Then, properties of the Schroedinger equation are summarized to prepare the reader for the main problem of the book, the numerical solution of the self-consistent field equations with and without exchange. The variation principle is carefully introduced, and the total energy of closed shell configurations discussed. Also, configurations comprising incomplete groups are treated. In the later part, the main ideas and methods are extended to more complicated or more complete

cases, and, in general, are described only very briefly. The text concludes with a chapter on "Better Approximations".

Many tables, relating to Slater coefficients, mean radii, screening numbers and reduced radial wave functions, are found in the text and in Appendix 2. For the bibliography up to 1947, the author refers to *Reports on Progress in Physics*, v. 11, 1946–47, p. 141–143, and completes the list to October 1956 in Appendices 1 and 3.

Numerical procedures, most of them recommended by the author's experience in hand and machine calculations, are described in detail, giving step-by-step instructions and numerical examples. The reviewer would have liked to have seen some comments on the numerical stability of methods using finite differences approximations to differential equations. Numerical stability is obvious in the Numerov and Fox-Goodwin process, but this is not so in Hartree's method of paragraph 4.6 (page 72), although the application is correct. Familiarity with numerical stability prevents the physicist from blindly refining the methods given in the text and will save costly "numerical experimentation".

In general, the book would gain by stating briefly the mathematical reasons why certain procedures are recommended (it would be mostly in the light of numerical stability!), as, for example, for the separation of integration of the radial wave equation into an outward and inward integration (§5.2). The physical reasons are stated adequately.

It seems that many equations were renumbered before the manuscript went to the printer. Cross references to equations are quite often unreliable. Otherwise, the number of misprints for a book of this kind is rather low.

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67[W, X].—ROBERT O. FERGUSON & LAUREN F. SARGENT, *Linear Programming*, McGraw-Hill Book Co., New York 1958, xiv + 342 p., 24 cm. Price $10.00.

With increasing frequency the professional mathematician, especially if he is working in applied mathematics, finds himself approached by friends or colleagues who lack advanced mathematical training but want to know more about the latest techniques, such as linear programming. In the course of the past year, several books on linear programming have appeared to which such inquirers might be referred.

This book should probably be regarded as the best of the group. It is addressed primarily to "people engaged in management activities at all levels in the firm and students of management. . . ." Its major virtues include a simple expository style without condescension, a wealth of illustrative examples, and a somewhat broader coverage of the subject than other works currently available. As a result of these qualities, it should prove suitable for individual study by management personnel with substantial practical experience in industry. It should, however, have its greatest value as a textbook for classroom instruction (on the job or off) of groups in which some or all participants lack the mathematical prerequisites which would permit use of a more advanced text, such as the well-known volume by Gass (from the same publishing house, interestingly enough).

The three sections into which the book is divided are entitled Introduction,

Methods, and Application, while two technical appendices treat the mathematical foundations of the Simplex Algorithm, and its relationships to the modi method, respectively. The second section presents the transportation method, the modi method, the simplex method, all of which are exact, and two approximation methods: the index method and the authors' own ratio-analysis method. Applications illustrated in detail in the third section include a product-mix problem, a production smoothing problem, and a problem in optimal assignment of orders to plants, where production costs as well as distribution costs affect the decision. Emphasis in this section is on obtaining and utilizing a maximal amount of information of value to management from the linear programming solution. Other applications, it may be added, are illustrated in presenting the computational methods in the second section. Further, two technical appendices treat the mathematical foundations of the Simplex Algorithm and its relationships to the modi method, respectively.

The professional mathematician should probably be warned that, while he may safely recommend this book to non-mathematicians, he should not attempt to read it himself, unless he is willing to take the risk of apoplexy. As evidence for this conclusion, one may cite a number of quite apoplectic reviews in which this book and others like it have been severely castigated (by mathematicians) for a low level of scholarship, lack of rigor, and similar mortal sins. To some, this attitude may seem unfair. Texts on business statistics, to take an analogy at random, are not usually criticized for omitting discussion of Borel sets, Stieltjes integrals, and the Cramer-Rao inequality. One might expect that there would be a place for a comparable treatment of linear programming without reference to theorems on matrix inversion or convex polyhedral cones.

For good or for ill, linear programming is being dished up for the common folk, and this book represents probably the most workmanlike presentation currently available. As might be expected, the book is weakest where it is most technical. The attempt, on page 50, to explain degeneracy explains nothing. Experience indicates that a better treatment of this concept is possible without resorting to advanced mathematics. Similarly, on page 5 and again on page 77, the difference between linear programming and the solution of simultaneous equations is explained in terms of the non-optimizing character of the latter, but the authors do not go on to explain, as they easily might have done, the reasons for this difference. Also, one of the examples (p. 119 ff.) used to illustrate the simplex method contains one redundant equation (because it is inherently a transportation problem) but the text makes no mention of this fact. As is all too often the case in technical works, the index is far from complete. For example, under "degeneracy" there is no reference to page 50, which has the only non-technical discussion of the topic (such as it is).

All these omissions could easily be remedied, and the many good qualities of the book warrant the hope that a future edition will see such improvements made.

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68[X].—Kurt Arbenz, *Integralgleichungen für einige Randwertprobleme für Gebiete mit Ecken*. Promotionsarbeit. Eidgenössische Technische Hochschule, Zürich, 1958, 43 p.

This paper is devoted to the problem of finding a procedure, suitable in numerical application, for the conformal representation of a simply connected plane domain over the unity circle, in the more difficult case of a boundary with corners. A useful tool which has been employed by Todd [1] is the integral equation of Lichtenstein and Gerschgorin, but it cannot be directly applied in this case. Modifications have been proposed by Stiefel and by Birkhoff, Young and Zarantonello [2].

The author generalizes theorems given by Radon, on the potential theory for domains bounded by smooth arcs. He makes extensive use of methods of functional analysis, with particular reference to the book of Riesz and Nagy [3].

The last seven pages contain numerical examples: the conformal representation of a square on the unit circle, obtained in a rather simple way with good accuracy; and the displacements in a square plate with built-in boundary. It appears that no use has been made of electronic computers, and it would be of interest to start numerical experiments on computers with the procedure here suggested.

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1. NBS APPLIED MATHEMATICS SERIES, No. 42, *Experiments in the Computation of Conformal Maps*, U. S. Government Printing Office, Washington, D. C., 1955.
2. *Proceedings of Symposia in Applied Mathematics*, v. 4, 1953, p. 117–140.
3. F. RIESZ & B. V. SZ. NAGY, *Leçons d'analyse fonctionelle*, Third Edition, Gauthier-Villars, Paris, 1955.

TABLE ERRATA

283.—G. W. SPENCELEY, R. M. SPENCELEY & E. R. EPPERSON, *Smithsonian Logarithmic Tables to Base e and Base* 10, Smithsonian Institution, Washington, D. C., 1952.

page	entry	*for*	*read*
2	ln 86	4.45435···	4.45434···
39	ln 1931	7.56475···	7.56579···
221	log 915	95142···	96142···
288	log 4271	63052 94714···	63052 95714···

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EDITORIAL NOTE: For additional errata, see *MTAC*, v. 10, 1956, p. 261; v. 11, 1957, p. 226.

J. W. W.

284.—E. CAMBI, *Eleven and Fifteen-Place Tables of Bessel Functions of the First Kind, to All Significant Orders*, Dover Publications, New York, 1948.

On p. 27, the value given for $J_0(9.76)$ should read $-.22836\cdots$ instead of $-.23836\cdots$.

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285.—W. MEYER ZUR CAPELLEN, *Integraltafeln. Sammlung unbestimmter Integrale elementarer Funktionen*, Springer, Berlin, 1950. [See RMT **1090**, *MTAC*, v. 7, 1953, p. 100.]

In addition to errata listed on a loose sheet supplied with the book and those appearing in MTE **223** (*MTAC*, v. 7, 1953, p. 106), the following corrections should be made:

p. 245, column 3, formula 1.5.1.1, line 1.

for $$\frac{\cos ka}{2a}[Ci\{k(a+x)\} + Ci\{k(a-x)\}]^*$$

read $$\frac{\cos ka}{2a}[Ci\{k(a+x)\} - Ci\{k(a-x)\}]^*.$$

p. 245, column 3, formula 1.5.1.2, line 1.

for $$-\frac{\cos ka}{2a}[Ci\{k(x+a)\} + Ci\{(x-a)\}]^*$$

read $$-\frac{\cos ka}{2a}[Ci\{k(x+a)\} - Ci\{k(x-a)\}]^*.$$

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NOTES

New Journals

Two new journals of interest to applied mathematicians have recently been announced.

Journal of Mathematical Analysis and Applications, published by Academic Press, New York and London. This new journal will provide a medium for the rapid publication of carefully selected mathematical papers treating classical analysis and its manifold applications.

The refereeing system used in most journals is replaced by a board of Associate Editors, each of whom may accept manuscripts. In this manner the delay between receipt and publication of manuscripts will be minimized. Only papers written in a lucid, expository style will be eligible for publication.

In recognition of the fact that other disciplines contribute new concepts and problems to the continuing growth of mathematics, papers devoted to the mathematical treatment of questions arising in physics, chemistry, biology, and engineering will be encouraged. In these papers the emphasis will be upon the analytical aspects and the novelty of problem and solution.

Journal of Mathematical Physics, published by the American Institute of Physics, New York, is a bimonthly devoted to new mathematical methods for the solution of physical problems as well as to original research in physics furthered by such methods. Its scope includes: mathematical aspects of quantum field theory, statistical mechanics of interacting particles, new approaches to eigenvalue and scattering problems, theory of stochastic processes, novel variational methods, theory of graphs, and review papers on mathematical topics for physicists.

Automatic Programming of Digital Computers—National Information Centre

The National Centre of Information on Automatic Programming of Digital Computers has been established by the Department of Mathematics of the Brighton Technical College, Brighton, England, in response to a recommendation of the first National Conference on Automatic Programming, held in Brighton in April, 1959. This conference was attended by 111 delegates from computer manufacturers, industrial and commercial computer users, government research establishments, universities and technical colleges.

The purpose of the Centre is:

(i) to establish and maintain a comprehensive library of publications, papers, research reports and other material, especially those not readily accessible, in any way relevant to the problems of automatic programming, and to make these available, in English, on demand;

(ii) to publish, in conjunction with Pergamon Press Limited, an annual review in Automatic Programming:

(iii) to provide a 'clearing house' for information and inquiries on automatic programming and related topics, and to help co-ordinate the work of other bodies active in this field;

(iv) to organize from time to time, small working conferences on particular aspects of the subject;

(v) to maintain permanent contact with organizations in all other countries concerned with such matters.

CORRIGENDUM

R. Hensman & D. P. Jenkins, *Tables of* $\frac{2}{\sqrt{\pi}} e^{z^2} \int_z^\infty e^{-t^2}\, dt$ *for Complex z, Math. Comp.*, Review 12, v. 14, 1960, p. 83.

for	read
$\frac{2}{\pi} e^{z^2} \int_z^\infty e^{-t^2}\, dt$	$\frac{2}{\sqrt{\pi}} e^{z^2} \int_z^\infty e^{-t^2}\, dt$

The research was done at the Royal Radar Establishment, Malvern, Worcestershire, England.

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